TOPIC 1
Lines and linear relationships

1.1 Overview

1.1.1 Introduction

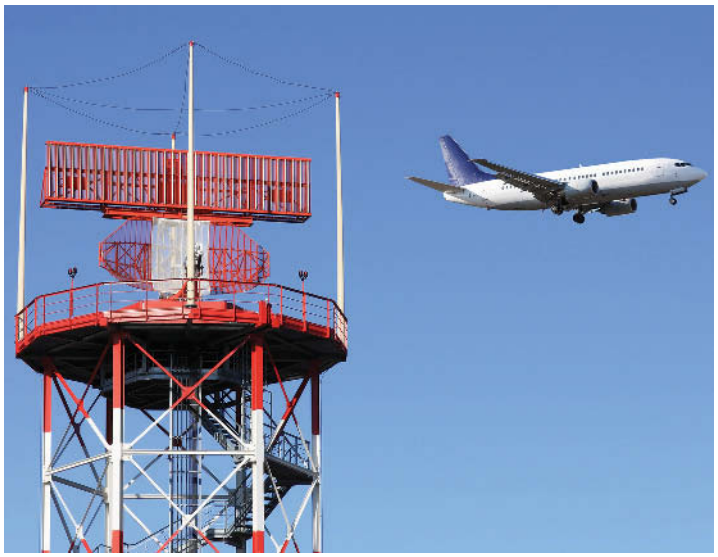
The French mathematician René Descartes (1596–1650) was one of the first to combine algebra and geometry.

It may be fictional, but it is said that while lying in bed one morning Descartes was engaged by the problem of how to describe the position of the fly that he was watching move about a wall in his bedroom. He proposed that the position of the fly could be fixed by specifying two numbers: one number giving the fly's distance from one wall and the other its distance from the adjoining perpendicular wall. The concept of specifying the position of a point using Cartesian coordinates had come to Descartes.

Furthermore, he recognised that algebra and geometry could be combined. Graphs, such as lines and circles, could be specified by equations. This branch of mathematics, developed by Descartes, Fermat and others, is called analytic, or coordinate geometry.

The idea of using a reference frame was not entirely new as the ancient Greeks had used such a concept. The astronomer Eudoxus (c.408–335 BCE) devised a complex coordinate system to represent the motion of the sun and moon with the Earth as origin.

Today coordinate systems are used in large-scale real-world applications including GPS tracking of vehicles from aircraft to delivery vans. GPS tracks positions relative to the two fixed axes of the Greenwich Prime Meridian and the equator. In the abstract, digital world, the three-dimensional polar and cylindrical coordinate systems are used in all computer animation.



LEARNING SEQUENCE

1.1 Overview
1.2 Linearly related variables, linear equations and inequations
1.3 Systems of 3 × 3 simultaneous linear equations
1.4 Linear graphs and their equations
1.5 Intersections of lines and their applications
1.6 Coordinate geometry of the straight line
1.7 Bisection and lengths of line segments
1.8 Review: exam practice

Fully worked solutions for this topic are available in the Resources section of your eBookPLUS at www.jacplus.com.au.

1.1.2 Kick off with CAS

Graphing straight lines with restrictions

1. Using CAS technology, graph the following straight lines.
   - **a.** $y = 2x$
   - **b.** $y = \frac{1}{2}x - 3$
   - **c.** $y = -x$
   - **d.** $y = -2x + 5$
2. The graphs of the linear relationships in question **1** stretch indefinitely in either direction. They are only limited by the boundaries of the screen. Using CAS technology, find the appropriate symbols and graph the following straight lines.
   - **a.** $y = 2x \,|\, 0 \leq x \leq 4$
   - **b.** $y = \frac{1}{2}x - 3 \,|\, -1 \leq x \leq 3$
   - **c.** $y = -x \,|\, x \leq 2$
   - **d.** $y = -2x + 5 \,|\, x \geq 0$
3. Using CAS technology, determine the endpoints of the straight lines in question **2**.
4. The image below is that of the famous Tower Bridge across the River Thames in London. Using the Cartesian plane and your knowledge of straight lines and parabolas, create a model of the image of the Tower Bridge using CAS technology.

on Resources

Please refer to the Resources section in the Prelims of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.

1.2 Linearly related variables, linear equations and inequations

For linearly related variables x and y, the rule connecting these variables is of the form $y = a + bx$. Plotting ordered pairs (x, y) which satisfy this relationship will create a straight-line graph.

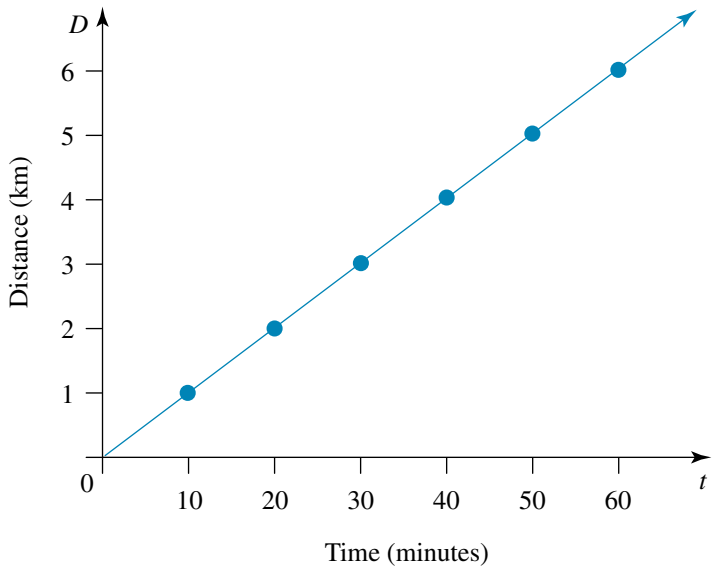
1.2.1 Variables in direct proportion

Two quantities which vary directly, or are in **direct proportion** or **direct variation**, have a linear relationship. For example, the distance travelled by a person jogging at a steady rate of 6 km/h (or 0.1 km/minute) would depend on the time spent jogging. The distance is in direct proportion to the time.

Time (minutes)	10	20	30	40	50	60
Distance (kilometres)	1	2	3	4	5	6

Doubling the time from 20 to 40 minutes causes the distance jogged to also double from 2 to 4 kilometres; trebling the time trebles the distance; halving the time halves the distance.

In this relationship, time is the **independent** or **explanatory variable** and distance the **dependent** or **response variable**. Plotting the points from the table on a distance versus time graph and joining these points gives a straight line through the origin. Here, time is the independent variable plotted along the horizontal axis, and distance is the dependent variable plotted along the vertical axis.



The linear relationship between these variables is described by the rule $D = 0.1t$, where t is the time in minutes and D the distance in kilometres.

For any variables x and y, if y is directly proportional to x:

- **then $y = kx$, where k is called the constant of proportionality**
- **the graph of y against x is a straight line through the origin.**

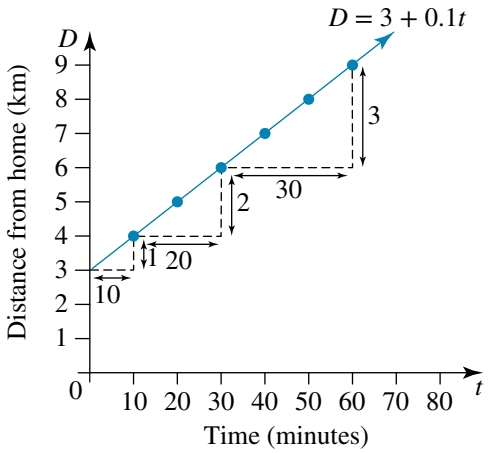
Other linearly related variables

Not all linear graphs pass through the origin, so not all linearly related variables are in direct proportion. However, intervals of change between the variables are in direct proportion.

If the jogger started from a point 3 km away from home, then the distance from home D kilometres after jogging for t minutes at the steady rate of 6 km/h (0.1 km/minute) would be given by $D = 3 + 0.1t$.

Although D is not directly proportional to t, if t changes by 10 minutes, then D changes by 1 km; if t changes by 20 minutes, D changes by 2 km; if t changes by 30 minutes, then D changes by 3 km; and so the changes in D are directly proportional to the changes in t. This occurs because the slope of the line is constant throughout.

The linear relationship $D = 3 + 0.1t$ can be described as the sum of two parts, one of which is a constant and the other of which is directly proportional to time. This is called a **part variation**.



WORKED EXAMPLE 1

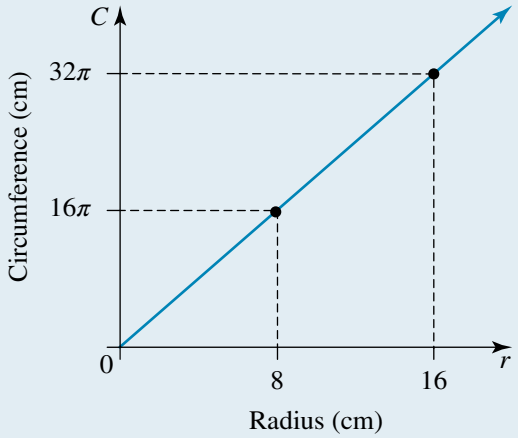
The circumference of a circle is directly proportional to its radius. If the circumference is 16π cm when the radius is 8 cm, calculate:

a. the constant of proportionality

b. the radius of the circle when the circumference is doubled and sketch the graph of circumference versus radius.

THINK	WRITE
a. 1. Define the two variables.	**a.** C = the length of the circumference in centimetres r = the length of the radius in centimetres
2. Write the direct variation statement using k for the proportionality constant.	$C = kr$
3. Substitute the given values and solve for k.	$C = 16\pi$ when $r = 8$ $16\pi = k(8)$ $k = \frac{16\pi}{8}$ $\therefore k = 2\pi$
b. 1. State the rule connecting C and r.	**b.** $C = 2\pi r$
2. Substitute the new value for C and solve for r.	Doubling the circumference $\Rightarrow C = 32\pi$ $32\pi = 2\pi r$ $\therefore r = 16$
3. Express the answer in context. *Note:* Doubling the circumference has doubled the radius.	The radius is 16 cm.

4. Sketch the graph.
r is the independent variable and C the dependent variable.
Since the circumference is directly proportional to the radius, the graph of C against r is a straight line through the origin.



on Resources

Interactivity: Linear relationships (int-2547)
Interactivity: Variables in direct proportion (int-2548)

1.2.2 Linear equations

Throughout this subject, the ability to solve equations and inequations of varying complexity will be required. Here we revise the underlying skills in solving linear equations, noting that a **linear equation** involves a variable which has an index or power of one. For example, $2x + 3 = 7$ is a linear equation in the variable x whereas $2x^2 + 3 = 7$ is not (it is a quadratic equation); and $2x + 3$ alone is an algebraic expression, not an equation.

WORKED EXAMPLE 2

Solve for x:
a. $4(3 + 2x) = 22 + 3x$ **b.** $\dfrac{2x + 3}{5} - \dfrac{1 - 4x}{7} = x.$

THINK

a. Expand the brackets and collect all terms in x together on one side in order to solve the equation.

b. 1. Express fractions with a common denominator and simplify.

WRITE

a. $4(3 + 2x) = 22 + 3x$
$12 + 8x = 22 + 3x$
$8x - 3x = 22 - 12$
$5x = 10$
$x = \dfrac{10}{5}$
$\therefore x = 2$

b. $\dfrac{2x + 3}{5} - \dfrac{1 - 4x}{7} = x$
$\dfrac{7(2x + 3) - 5(1 - 4x)}{35} = x$
$\dfrac{14x + 21 - 5 + 20x}{35} = x$
$\dfrac{34x + 16}{35} = x$

2. Remove the fraction by multiplying both sides by the common denominator and solve for x.

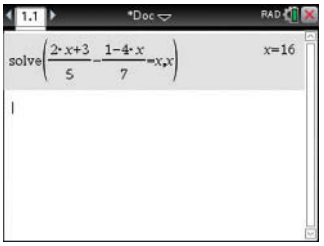
$$34x + 16 = 35x$$
$$16 = 35x - 34x$$
$$\therefore x = 16$$

TI | THINK

b. 1. On a Calculator page, press MENU and select:
3. Algebra
1. Solve
Complete the entry line as: solve $\left(\frac{2x+3}{5} - \frac{1-4x}{7} = x, x\right)$
Then press ENTER.

2. The answer appears on the screen.

WRITE



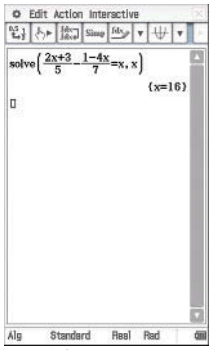
$x = 16$

CASIO | THINK

b. 1. On the Main screen, complete the entry line as:
solve $\left(\frac{2x+3}{5} - \frac{1-4x}{7} = x, x\right)$
Then press EXE.

2. The answer appears on the screen.

WRITE



$x = 16$

Literal linear equations

Literal equations contain pronumerals rather than known numbers. The solution to a literal equation in x usually expresses x as a combination of these pronumerals rather than as a specific numerical value. Although the method of solution of linear literal equations is similar to those previously used, it is worth checking to see if answers may be simplified using algebraic skills such as factorisation.

WORKED EXAMPLE 3

Solve for x: $\frac{x+a}{b} = \frac{b-x}{a}$.

THINK

1. Each fraction can be placed on the common denominator and then each side multiplied by that term.
Note: Since there is only one fraction on each side of this equation, a quick way to do this is to 'cross-multiply'.

2. Collect all the terms in x together and take out x as the common factor.

3. Divide by the coefficient of x to obtain an expression for x.

4. Simplify the expression, if possible.

WRITE

$$\frac{x+a}{b} = \frac{b-x}{a}$$
$$a(x+a) = b(b-x)$$
$$ax + a^2 = b^2 - bx$$

$$ax + bx = b^2 - a^2$$
$$x(a+b) = b^2 - a^2$$
$$\therefore x = \frac{b^2 - a^2}{a+b}$$

The numerator can be factorised as a difference of two squares.

$$x = \frac{b^2 - a^2}{a+b}$$
$$= \frac{(b-a)(b+a)}{a+b}$$

5. Cancel the common factor to give the solution in its simplest form.

$$x = \frac{(b-a)\cancel{(b+a)}}{\cancel{a+b}}$$
$$\therefore x = b - a$$

1.2.3 Linear inequations

An **inequation** contains one of the order symbols:

<(less than), ≤ (less than or equal to), > (greater than), ≥ (greater than or equal to).

The order symbols are referred to as inequality symbols.

Linear inequations are solved in a similar way to linear equations; however, care must be taken when multiplying or dividing by a negative number.

Remember to reverse the order symbol when multiplying or dividing by a negative number.

To illustrate this, consider the inequality statement that $-6 < 15$. If this inequality is divided by -3 then the statement must become $2 > -5$, so the order symbol has been reversed.

The solutions to linear inequations are sets of values satisfying an inequality, unlike linear equations where the solutions are unique.

When illustrating inequalities on a number line:

- an open circle, ○, is used when the endpoint is not included (< or >)
- a closed circle, •, is used when the endpoint is included (≤ or ≥).

WORKED EXAMPLE 4

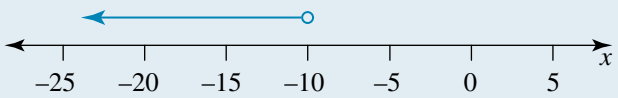
Calculate the values for x for which $5 - \frac{4x}{5} > 13$ and show this set of values on a number line.

THINK

1. Subtract 5 from both sides of the inequation.
   Note: Subtracting a number does not affect the inequality symbol.
2. Multiply both sides by 5.
   Note: Multiplying by a positive number does not affect the inequality symbol.
3. Divide both sides by -4.
   Note: Dividing by a negative number does require the symbol to be reversed.
4. Illustrate this set of values on a number line.

WRITE

$$5 - \frac{4x}{5} > 13$$
$$-\frac{4x}{5} > 13 - 5$$
$$-\frac{4x}{5} > 8$$
$$-4x > 8 \times 5$$
$$-4x > 40$$
$$x < \frac{40}{-4}$$
$$\therefore x < -10$$



The number line has an open end at $x = -10$ since this value is not included in the set of solutions, $x < -10$.

1.2.4 Systems of 2 × 2 simultaneous linear equations

To find the value of one variable, one equation is needed. However, to find the values of two variables, two equations are required. These two equations form a 2×2 **system of equations**.

The usual algebraic approach for finding the values of two variables, x and y, which satisfy two linear equations simultaneously is by using either a ***substitution*** method or an ***elimination*** method.

Substitution method

In the substitution method, one equation is used to express one variable in terms of the other and this expression is substituted in place of that variable in the second equation. This creates an equation with just one variable which can then be solved.

Elimination method

In the elimination method the two equations are combined in such a way as to eliminate one variable, leaving an equation with just one variable which can then be solved.

WORKED EXAMPLE 5

a. Use the substitution method to solve the following system of simultaneous equations for x and y.

$$y = 3x - 1$$
$$x - 3y = 11$$

b. Use the elimination method to solve the following system of simultaneous equations for x and y.

$$8x + 3y = -13$$
$$5x + 4y = -6$$

THINK	WRITE
a. 1. Write the two equations and number them for ease of reference.	**a.** $y = 3x - 1$ [1] $x - 3y = 11$ [2]
2. Since the first equation already has y in terms of x, it can be substituted for y in the second equation.	Substitute equation [1] into equation [2]. $x - 3(3x - 1) = 11$
3. Solve for x.	$x - 9x + 3 = 11$ $-8x = 11 - 3$ $-8x = 8$ $\therefore x = -1$
4. Substitute the value of x into one of the original equations and calculate the y-value.	Substitute $x = -1$ into equation [1]. $y = 3(-1) - 1$ $\therefore y = -4$
5. State the answer.	The solution is $x = -1$ and $y = -4$.
6. It is a good idea to check that these values satisfy both equations by substituting the values for x and y into the other equation.	Check: In equation [2], if $x = -1$ and $y = -4$, then: $x - 3y = (-1) - 3 \times (-4)$ $= -1 + 12$ $= 11$ True

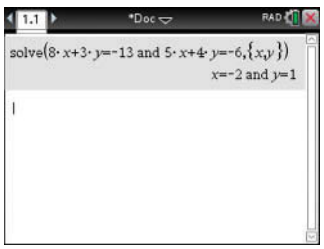
b. 1. Write the two equations and number them for ease of reference.	**b.** $8x + 3y = -13$ [1] $5x + 4y = -6$ [2]
2. Choose whether to eliminate x or y and adjust the coefficient of the chosen variable in each equation. *Note:* Neither equation readily enables one variable to be expressed simply in terms of the other, so the elimination method is the more appropriate method here.	Eliminate y. Multiply equation [1] by 4: $32x + 12y = -52$ [3] Multiply equation [2] by 3: $15x + 12y = -18$ [4]
3. Eliminate y and solve for x. *Note:* 'Same Signs Subtract' can be remembered as the SSS rule.	Equation [3] − equation [4] $32x - 15x = -52 - (-18)$ $17x = -34$ $\therefore x = -2$
4. Substitute the value of x into one of the original equations and calculate the y-value. *Note:* Alternatively, start again but this time eliminate x.	Substitute $x = -2$ into equation [2] $5(-2) + 4y = -6$ $-10 + 4y = -6$ $4y = -6 + 10$ $4y = 4$ $\therefore y = 1$
5. State the answer.	The solution is $x = -2, y = 1$.

TI | THINK

b. 1. On a Calculator page, press MENU and select:
3. Algebra
1. Solve
Complete the entry line as:
solve
$(8x + 3y = -13$ and $5x + 4y = -16, \{x, y\})$
Then press ENTER.

2. The answer appears on the screen.

WRITE



$x = -2, y = 1$

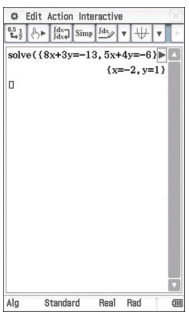
CASIO | THINK

b. 1. On the Main screen, select:
- Action
- Advanced
- solve

Complete the entry line as:
solve $(\{8x + 3y = -13, 5x + 4y = -6\}, \{x, y\})$
Then press EXE.

2. The answer appears on the screen.

WRITE



$x = -2, y = 1$

on Resources

Interactivity: Simultaneous equations (int-2549)

1.2.5 Solving problems using equations

The symbolism of algebra enables statements of problems to be expressed very concisely and succinctly. This is part of its power and beauty. Mathematicians may construct sets of equations to model relationships between various variables and, by solving these, contribute to our collective understanding of real-world phenomena.

In setting up equations:

- define the symbols used for the variables, specifying units where appropriate
- ensure any units used are consistent
- express answers in the context of the problem.

WORKED EXAMPLE 6

The organisers of an annual student fundraising event for charity know there will be fixed costs of $120 plus an estimated cost of 60 cents per student for incidental costs on the day of the fundraiser. The entry fee to the fundraising event is set at $5.

a. Form an algebraic model for the profit the event can expect to make.

b. What is the least number of students who must attend the event to avoid the organisers making a loss?

A proposal is made to lower the student entry fee by allowing staff to attend the event.

c. If 30 students and 5 staff attend, the revenue gained on entry fees is $145 whereas if 55 students and 15 staff attend the revenue is $312.50. How much is the proposed entry fee for a student and how much is it for a member of staff?

THINK	WRITE
a. 1. Define the variables.	**a.** Let n = the number of students attending the event P = the profit made in dollars
2. Form expressions for the cost and the revenue, ensuring units are consistent.	Profit depends on costs and revenues. Revenue ($) = $5n$ Costs ($) = $120 + 0.60n$
3. Form the expression for the profit to define the algebraic model.	Profit = revenue − costs Hence, $P = 5n - (120 + 0.60n)$ $\therefore P = 4.4n - 120$ gives the linear model for the profit.
b. 1. Impose the condition required to avoid a loss and calculate the consequent restriction on n.	**b.** A loss is made if $P < 0$. To avoid making a loss, the organisers require $P \geq 0$. $4.4n - 120 \geq 0$ $4.4n \geq 120$ $n \geq \frac{120}{4.4}$ $\therefore n \geq 27.2727...$
2. Express the answer in the context of the question.	To avoid making a loss, at least 28 students need to attend the event.
c. 1. The problem is now posed in terms of other variables. Define these and express the given information using a pair of simultaneous equations.	**c.** Let the entry fees be: s dollars for students and t dollars for staff. $30s + 5t = 145$ [1] $55s + 15t = 312.5$ [2]
2. Solve this system of simultaneous equations by elimination.	Eliminate t. Multiply equation [1] by 3. $90s + 15t = 435$ [3] $55s + 15t = 312.5$ [2]

Subtract equation [2] from equation [3].

$90s - 55s = 435 - 312.5$

$35s = 122.5$

$\therefore s = \frac{122.5}{35}$

$\therefore s = 3.5$

Substitute $s = 3.5$ into equation [1].

$30(3.5) + 5t = 145$

$105 + 5t = 145$

$5t = 40$

$\therefore t = 8$

3. Write the answer in context.

The student entry fee is \$3.50 and the staff entry fee is \$8.

studyon

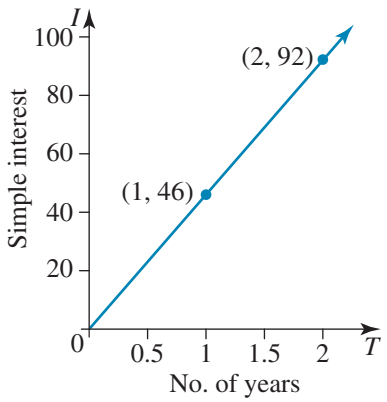
Units 1 & 2 > AOS 1 > Topic 1 > Concept 1

Linearly related variables Summary screen and practice questions

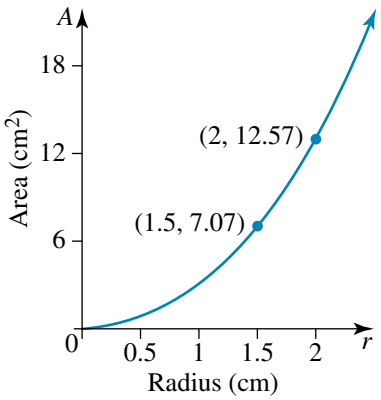
Exercise 1.2 Linearly related variables, linear equations and inequations

Technology free

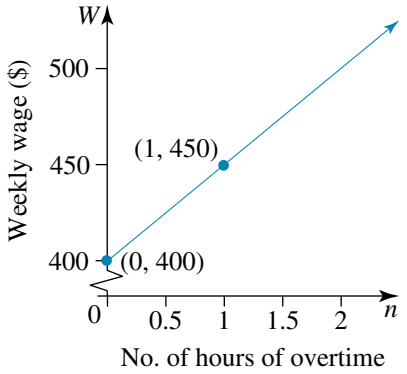
1. **WE1** The volume of a cone of fixed base radius is directly proportional to its height. If the volume is 96π cm^3 when the height is 6 cm, calculate:
   a. the constant of proportionality
   b. the height of this cone when its volume is halved and sketch the graph of volume versus height.
2. From the following diagrams **a** to **c**, select those graphs which show two linearly related variables. For these, describe the relationship and determine the linear rule between the variables.
   a. The simple interest (I) earned on an investment of \$100 versus the number of years (T) of investment



b. The area (A) of a circle, in cm^2, versus the length of the radius (r), in cm



c. The weekly wage earned (W) versus the number of hours (n) of overtime work



3. **WE2** Solve for x:

a. $3(5x - 1) = 4x - 14$

b. $\frac{4 - x}{3} + \frac{3x - 2}{4} = 5$

4. Solve the following linear equations

a. $2x - 5 = 11$

b. $\frac{x}{2} + 4 = -1$

c. $2x - 4 = 5x + 3$

d. $5x + 3 = -4(1 - x)$

e. $-1 + \frac{5x + 2}{3} = 10$

f. $\frac{3 - 2x}{6} = \frac{x + 4}{4}$

g. $\frac{3x - 2}{4} + \frac{x + 4}{5} = -2$

h. $\frac{4x - 5}{6} - \frac{x + 4}{12} = 1$

5. Solve the following linear equations for x.

a. $7(2x - 3) = 5(3 + 2x)$

b. $\frac{4x}{5} - 9 = 7$

c. $4 - 2(x - 6) = \frac{2x}{3}$

d. $\frac{3x + 5}{9} = \frac{4 - 2x}{5}$

e. $\frac{x + 2}{3} + \frac{x}{2} - \frac{x + 1}{4} = 1$

f. $\frac{7x}{5} - \frac{3x}{10} = 2\left(x + \frac{9}{2}\right)$

6. The cost per day of running a children's crèche is the sum of two quantities, one of which is the constant fixed cost of $210 and the other of which is directly proportional to the number of children attending the crèche.
If the cost is $330 on a day when 20 children attend:

a. form the linear relationship between the cost per day, C dollars, and n, the number of children attending the crèche that day.

b. What will the cost be on a day when 25 children attend?

7. Solve each of the following for m by making m the subject.

a. $\frac{m}{8} + z = x$ **b.** $b - \frac{a}{m} = cd$ **c.** $2m + c = 3d - m$

d. $md - k = d + mc$ **e.** $\frac{2}{m} + \frac{3}{b} = c$ **f.** $\frac{2m - 3}{d} = \frac{4 - m}{a}$

8. **WE3** Solve for x: $\frac{d - x}{a} = \frac{a - x}{d}$

9. Solve the following literal equations for x, expressing solutions in simplest form.

a. $ax + b = c$ **b.** $a(x - b) = bx$ **c.** $a^2x + a^2 = ab + abx$

d. $\frac{x}{a} + \frac{x}{b} = a + b$ **e.** $\frac{bx - a}{c} = \frac{cx + a}{b}$ **f.** $\frac{x + a}{b} - 2 = \frac{x - b}{a}$

10. **WE4** Calculate the values for x for which $7 - \frac{3x}{8} \leq -2$ and show this set of values on a number line.

11. Solve the following inequations

a. $4 - 2x \geq 5$ **b.** $\frac{x - 6}{3} + 4 < 1$ **c.** $2x - 3 < 4x + 1$

d. $-2(x - 5) - x > 3(x + 4)$ **e.** $1 - \frac{4 - x}{2} > -1$ **f.** $\frac{5 - x}{2} < -\frac{3x + 2}{8}$

12. Solve the following inequations for x.

a. $3x - 5 \leq -8$ **b.** $4(x - 6) + 3(2 - 2x) < 0$

c. $1 - \frac{2x}{3} \geq -11$ **d.** $\frac{5x}{6} - \frac{4 - x}{2} > 2$

e. $8x + 7(1 - 4x) \leq 7x - 3(x + 3)$ **f.** $\frac{2}{3}(x - 6) - \frac{3}{2}(x + 4) > 1 + x$

13. **WE5** **a.** Use the substitution method to solve the following system of simultaneous equations for x and y.

$$x = 2y + 5$$
$$4x - 3y = 25$$

b. Use the elimination method to solve the following system of simultaneous equations for x and y.

$$5x + 9y = -38$$
$$-3x + 2y = 8$$

14. Solve each of the following sets of simultaneous equations for x and y.

a. $4x + 3y = 5$
$y = x - 3$

b. $x = 2y - 4$
$x = 1 - 8y$

c. $7x - 3y = 11$
$2x + 3y = 7$

d. $2x + 3y = 10$
$x - y = -5$

e. $2x + 3y = 11$
$3x + 5y = 18$

f. $4x - 3y = -38$
$5x + 2y = -13$

15. Solve the following simultaneous equations for x and y.

a. $y = 5x - 1$
$x + 2y = 9$

b. $3x + 5y = 4$
$8x + 2y = -12$

c. $x = 5 + \frac{y}{2}$
$-4x - 3y = 35$

d. $8x + 3y = 8$
$-2x + 11y = \frac{35}{6}$

e. $\frac{x}{2} + \frac{y}{3} = 8$
$\frac{x}{3} + \frac{y}{2} = 7$

f. $ax + by = 4ab$
$ax - by = 2ab$

Technology active

16. Solve for x: $\frac{7(x-3)}{8} + \frac{3(2x+5)}{4} = \frac{3x}{2} + 1$

17. Solve for x: $b(x + c) = a(x - c) + 2bc$

18. Solve for x: $4(2 + 3x) > 8 - 3(2x + 1)$

19. Solve for x and y:

a. $2x - y = 7$
$7x - 5y = 42$

b. $ax - by = a$
$bx + ay = b$

20. Use CAS technology to solve:

a. $3(5x - 2) + 5(3x - 2) = 8(x - 2)$

b. $\frac{2x-1}{5} + \frac{3-2x}{4} < \frac{3}{20}$

c. $4x - 3y = 23$
$7x + 4y = 31$

d. $3(x + 2) = 2y$
$7x - 6y = 146$

21. Form the linear equation connecting the two variables and then use the equation to answer the question in each of the following:

a. When an elastic string is extended x units beyond its natural length, the tension T in the string is directly proportional to the extension. When the string is extended by 0.2 metres the tension is 0.7 newtons. What is the tension in the string after it has been extended a further 0.1 metre?

b. The cost C of purchasing petrol from a petrol station is directly proportional to the number of litres l poured into the car's tank. If it costs \$52.49 to fill a tank with 36.2 litres, how many litres of petrol can be purchased for \$43.50?

c. A ball is thrown vertically upwards with a velocity v of 12 m/s, which then decreases by an amount proportional to t, the number of seconds the ball moves upwards. After 0.5 seconds the ball has slowed to a velocity of 7.1 m/s. How many seconds after it is thrown upwards will the ball start to fall back to the ground?

22. Solve the following problems by first forming linear equations.

a. The sum of two consecutive even numbers is nine times their difference. What are these two numbers?

b. Three is subtracted from a certain number and the result is then multiplied by 4 to give 72. What is the number?

c. The sum of three consecutive numbers is the same as the sum of 36 and one quarter of the smallest number. What are the three numbers?

d. The length of a rectangle is 12 cm greater than twice its width. If the perimeter of the rectangle is 48 cm, calculate its length and width.

e. The ratio of the length to the width to the height of a rectangular prism is 2:1:3 and the sum of the lengths of all its edges is 360 cm. Calculate the height of this rectangular prism.

23. WE6 Although the organisers of a secondhand book sale are allowed free use of the local Scouts Hall for their fete, they must contribute $100 towards heating and lighting costs and in addition donate 20*c* from the sale of each book to the Scouts Association. The books are intended to be sold for $2.50 each.

a. Form an algebraic model for the profit the book sale can expect to make.

b. What is the least number of books that must be sold to ensure the organisers make a profit?

A proposal is made to sell the hardcover books at a different price to the paperback books.

c. If 40 hardback books and 65 paperbacks are sold, the revenue gained from the sales would be $330 whereas the sale of 30 hardback books and 110 paperbacks would bring revenue of $370. How much are the sale prices of hardback books and of paperback books under this proposal?

24. a. When 4 adults and 5 children attend a pantomime, the total cost of the tickets is $160 whereas the cost of the tickets for 3 adults and 7 children is $159. How much is an adult ticket and how much does a child's ticket cost?

b. A householder's electricity bill consists of a fixed payment together with an amount proportional to the number of units used. When the number of units used was 1428 the total bill was $235.90 and when the number of units was 2240, the bill was $353.64. How much will the householder's bill be if 3050 units are used?

c. The temperature measured in degrees Celsius, C, is linearly related to the temperature measured in degrees Fahrenheit (F). Water boils at 100 °C or 212 °F and freezes at 0 °C or 32 °F. Find the linear equation that enables conversion of degrees Fahrenheit to degrees Celsius.

25. A cyclist travels at an average speed of 16 km/h along a road. Fifteen minutes after the cyclist sets out, a motorcyclist starting from the same place travels along the same road at an average speed of 48 km/h. How many minutes will the cyclist have travelled before being overtaken by the motorcyclist?

26. a. A car travelling at 60 km/h takes t hours to go from A to B. If the speed of the car is reduced by 10 km/h, the time to go from A to B is increased by half an hour. Calculate the value of t and find the distance between A and B.

b. A cyclist travelling at an average speed of u km/h takes t hours to travel from A to B. On reaching B, the cyclist rests for $0.1t$ hours before returning from B to A along the same route. If the cyclist reaches A $2t$ hours after first setting out, find an expression for the average speed of the cyclist on the return journey.

27. A cricket social club committee booked four tables at a local restaurant for a casual lunch for their members. The organisers were unsure in advance how many people were attending but on the day three or four people sat at each table.

a. Construct an inequality to describe the number of people from the cricket club who actually attended the lunch.

b. The lunch consisted of two courses for a fixed amount per person. If the total bill the cricket social club had to pay was exactly $418.50, how many people from the cricket club attended the lunch and what was the fixed charge per person?

c. Included in the lunch cost was complimentary tea or coffee. If the number of tea drinkers was half the number of coffee drinkers and each person had either a coffee or a tea, how many of the cricket club members drank coffee and how many drank tea?

28. a. It is thought that the chirping rate of cicadas is linearly related to the temperature. Use the given data to construct a linear model of the chirping rate C and temperature T.

Temperature (°C)	Chirping rate (chirps/minute)
21°	113
27°	173

b. Solve for x and y:

$$ax + bcy = c$$
$$bx + acy = c.$$

1.3 Systems of 3 × 3 simultaneous linear equations

The yield from a crop of zucchini plants appears to be linearly related to the amount of sunshine and the amount of rainfall the crop receives. This could be expressed as a linear equation $z = a+bx+cy$ using the variables z for the number of zucchini harvested, x for the amount of sunshine and y for the amount of rain. Such linear relationships have three variables, x, y and z, so their graph would require a sketch in 3-dimensional space (something for future mathematical studies).

However, situations may arise in which we need to solve a system of three simultaneous linear equations in three unknowns. It will be sufficient to restrict our attention to solving systems that have unique solutions (although this is not always the case for such systems).

on Resources

Interactivity: Equations in three variables (int-2550)

1.3.1 The manual method for solving 3 equations in 3 variables

The steps involved in solving 3 equations in 3 variables are as follows:

- Reduce the 3 × 3 system to a 2 × 2 system by eliminating the same variable from two different pairs of equations.
- Solve the 2 × 2 system of simultaneous equations using an elimination or substitution method to obtain the values of these two variables.
- Substitute these values for two of the variables into an original equation to obtain the value of the third variable.

Where one equation allows a variable to easily be expressed in terms of the other two variables, substituting this expression into both of the other equations would create the 2 × 2 system of simultaneous equations.

Note: A CAS calculator or other technology may be used to solve 3 × 3 systems of equations.

WORKED EXAMPLE 7

Solve for x, y and z in the following 3×3 system of simultaneous equations:

$$3x - 4y + 5z = 10$$
$$2x + y - 3z = -7$$
$$5x + y - 2z = -9$$

THINK	WRITE
1. Number the equations for ease of reference.	$3x - 4y + 5z = 10$ [1] $2x + y - 3z = -7$ [2] $5x + y - 2z = -9$ [3]
2. Decide which variable to eliminate and choose a pair of equations to carry out this elimination. Number the resultant equation.	Eliminate y from equations [1] and [2]. Multiply equation [2] by 4. $8x + 4y - 12z = -28$ [4] $3x - 4y + 5z = 10$ [1] Add equations [4] and [1]. $\therefore 11x - 7z = -18$ [5]
3. Choose a second pair of equations to eliminate the same variable. Number the resultant equation.	Eliminate y from equations [3] and [2]. $5x + y - 2z = -9$ [3] $2x + y - 3z = -7$ [2] Subtract equation [2] from equation [3]. $\therefore 3x + z = -2$ [6]
4. The original 3×3 system has been reduced to a 2×2 system of simultaneous equations in x and z. Write these equations.	$11x - 7z = -18$ [5] $3x + z = -2$ [6]
5. Solve the equations to find x and z. *Note:* Either of the elimination or substitution methods could be used. Looking at the coefficients of z in each of the two equations, we shall choose to eliminate z.	Eliminate z. Add equation [5] and 7 times equation [6]. $11x + 7 \times 3x = -18 + 7 \times (-2)$ $32x = -32$ $x = -\frac{32}{32}$ $\therefore x = -1$ Substitute $x = -1$ into equation [6]. $-3 + z = -2$ $\therefore z = 1$
6. Finally, to obtain y, substitute the values of x and z into one of the original equations. *Note:* For ease of calculation, it is usual to select the equation which looks the simplest.	Substitute $x = -1$, $z = 1$ into equation [2]. $2(-1) + y - 3(1) = -7$ $-2 + y - 3 = -7$ $y - 5 = -7$ $\therefore y = -2$
7. State the answer.	The solution is $x = -1$, $y = -2$ and $z = 1$.
8. It is a good idea to check that the solution satisfies the other two equations by substituting the values and showing that LHS (left-hand side) equals RHS (right-hand side).	Check: In equation [1] substitute $x = -1$, $y = -2$ and $z = 1$. LHS $= 3 \times (-1) - 4 \times (-2) + 5 \times (1)$ $= -3 + 8 + 5$ $= 10$ $=$ RHS

In equation [3] substitute $x = -1$, $y = -2$ and $z = 1$.

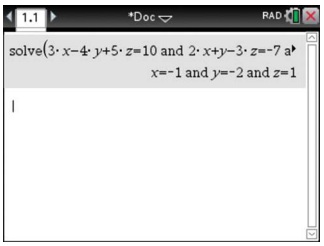
$$\begin{aligned}\text{LHS} &= 5 \times (-1) + (-2) - 2 \times (1) \\ &= -5 - 2 - 2 \\ &= -9 \\ &= \text{RHS}\end{aligned}$$

TI | THINK

b. 1. On a Calculator page, press MENU and select:
3. Algebra
1. Solve
Complete the entry line as:
solve $(3x - 4y + 5z = 10$ and $2x + y - 3z = -7$ and $5x + y - 2z = -9, \{x, y, z\})$
Then press ENTER.

2. The answer appears on the screen.

WRITE



$x = -1, y = -2, z = 1$

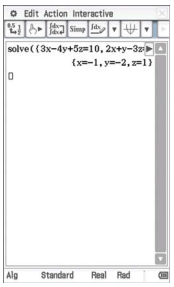
CASIO | THINK

b. 1. On the Main screen, select:
- Action
- Advanced
- solve

Complete the entry line as:
solve $(3x - 4y + 5z,$
$2x + y - 3z = -7,$
$5x + y - 2z = -9$, $\{x, y, z\})$
Then press EXE.

2. The answer appears on the screen.

WRITE



$x = -1$, $y = -2$, $z = 1$

study on

Units 1 & 2 > AOS 1 > Topic 1 > Concept 2

Systems of simultaneous linear equations Summary screen and practice questions

Exercise 1.3 Systems of 3 × 3 simultaneous linear equations

Technology free

1. Consider the 3 × 3 system of simultaneous equations:

$$\begin{aligned} x + 2y + z &= 6 \quad [1] \\ 4x - y - z &= 7 \quad [2] \\ 3x + 4y + z &= 8 \quad [3] \end{aligned}$$

a. i. Which variable would be eliminated by adding together equations [1] and [2]?
ii. Explain how z could be eliminated from equations [1] and [3].

b. MC Which of the following operations would create an equation in which y was eliminated?
A. [3] − [2] **B.** 2 × [3] + [1] **C.** 2 × [3] − [1] **D.** 2 × [2] + [1] **E.** [1] − 2 × [2]

c. Equation [4] is formed by adding equations [2] and [3]. Write down equation [4].

d. Equation [5] is formed by adding equations [1] and [2]. Write down equation [5].

e. Show that $x = 3$ and $y = -2$ satisfy both equations [4] and [5].

f. Calculate the value of z.

2. Consider the 3 × 3 system of equations shown.

$$\begin{aligned} 5x + 4y + 2z &= 2 \quad [1] \\ 2x + 3y + 2z &= -8 \quad [2] \\ -7x + 2y + 2z &= -42 \quad [3] \end{aligned}$$

To solve for x, y and z, first reduce to a 2×2 system by eliminating the same variable from two different pairs of the equations. Suppose we choose to eliminate z.

a. Copy the following and complete the working in the spaces shown to obtain equations [4] and [5]

$5x + 4y + 2z = 2$	[1]	$2x + 3y + 2z = -8$	[2]
$2x + 3y + 2z = -8$	[2]	$-7x + 2y + 2z = -42$	[3]
equation [1] − [2]		equation [2] − [3]	
.....	[4]	.....	[5]

b. Solve equations [4] and [5] to find x and y

c. Substitute these values for x and y into one of the original equations, to calculate z.

3. **WE7** Solve for x, y and z in the following 3×3 system of simultaneous equations:

$$5x - 2y + z = 3$$
$$3x + y + 3z = 5$$
$$6x + y - 4z = 62.$$

4. Solve for x, y and z in the following 3×3 system of simultaneous equations:

$$z = 12 - x + 4y$$
$$z = 4 + 5x + 3y$$
$$z = 5 - 12x - 5y.$$

In questions **5–10**, solve the 3×3 systems of simultaneous equations for x, y and z.

5.
$$2x + 3y - z = 3$$
$$5x + y + z = 15$$
$$4x - 6y + z = 6$$

6.
$$x - 2y + z = -1$$
$$x + 4y + 3z = 9$$
$$x - 7y - z = -9$$

7.
$$2x - y + z = -19$$
$$3x + y + 9z = -1$$
$$4x + 3y - 5z = -5$$

8.
$$2x + 3y - 4z = -29$$
$$-5x - 2y + 4z = 40$$
$$7x + 5y + z = 21$$

9.
$$3x - 2y + z = 8$$
$$3x + 6y + z = 32$$
$$3x + 4y - 5z = 14$$

10.
$$y = 3x - 5$$
$$\frac{x - z}{2} - y + 10 = 0$$
$$9x + 2y + z = 0$$

Technology active

In questions **11–14**, set up a system of simultaneous equations to use to solve the problems.

11. At a recent art exhibition the total entry cost for a group of 3 adults, 2 concession holders and 3 children came to $96; 2 adult, 1 concession and 6 child tickets cost $100; and 1 adult, 4 concession and 1 child ticket cost $72. Calculate the cost of an adult ticket, a concession ticket and a child ticket.

12. Agnes, Bjork and Chi are part-time outsource workers for the manufacturing industry. When Agnes works 2 hours, Bjork 3 hours and Chi 4 hours, their combined earnings total $194. If Agnes works 4 hours, Bjork 2 hours and Chi 3 hours, their total earnings are $191; and if Agnes works 2 hours, Bjork 5 hours and Chi 2 hours their combined earnings total $180. Calculate the hourly rate of pay for each person.

13. A nutritionist at a zoo needs to produce a food compound in which the concentration of fats is 6.8 kg unsaturated fats, 3.1 kg saturated fats and 1.4 kg trans fats. The food compound is formed from three supplements whose concentration of fats per kg is shown in the following table:

	Unsaturated fat	Saturated fat	Trans fat
Supplement X	6%	3%	1%
Supplement Y	10%	4%	2%
Supplement Z	8%	4%	3%

How many kilograms of each supplement must be used in order to create the food compound?

14. A student buys a sandwich at lunchtime from the school canteen for $4.20 and pays the exact amount using 50 cent coins, 20 cent coins and 10 cent coins. If the number of 20 cent coins is the same as half the 10 cent coins plus four times the number of 50 cent coins and the student pays the cashier with 22 coins in total, how many coins of each type did the student use?

15. a. Use CAS technology to obtain the values of x, y and z for the system of equations:

$$\begin{aligned} 2x + 6y + 5z &= 2 \\ 5x - 10y - 8z &= 20.8 \\ 7x + 4y + 10z &= 1 \end{aligned}$$

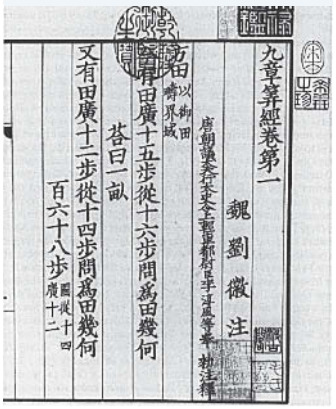
b. Use CAS technology to obtain the values of x, y, z and w for the 4×4 system of equations:

$$\begin{aligned} x - y + 4z - 2w &= 8 \\ 3x + 2y - 2z + 10w &= 67 \\ 2x + 8y + 18z + w &= -14 \\ 8x - 7y - 80z + 7w &= 117 \end{aligned}$$

c. Use CAS technology to obtain the values of x_1, x_2, x_3 and x_4 for the 4×4 system of equations:

$$\begin{aligned} 4x_1 + 2x_2 + 3x_3 + 6x_4 &= -13 \\ 12x_1 - 11x_2 - 7.5x_3 + 9x_4 &= 16.5 \\ x_1 + 18x_3 - 12x_4 &= 8 \\ -3x_1 + 12x_2 - x_3 + 10x_4 &= -41 \end{aligned}$$

Earlier than the Europeans, possibly around 250 BC, the work of Chinese mathematicians in *Chui-chang suan-shu* (*Nine chapters on the Mathematical Art*), gave the solution to simultaneous linear equations in 3 unknowns.



16. The yield of zucchini, z kg/hectare, over a period of x hours of sunshine and y mm of rainfall is shown in the following table.

z	x	y
23	30	320
28	50	360
30	40	400

a. Form a linear model in the form $z = a + bx + cy$ for this data.

b. How many kilograms per hectare of zucchini would the model predict for 40 hours of sunshine and only 200 mm of rain?

1.4 Linear graphs and their equations

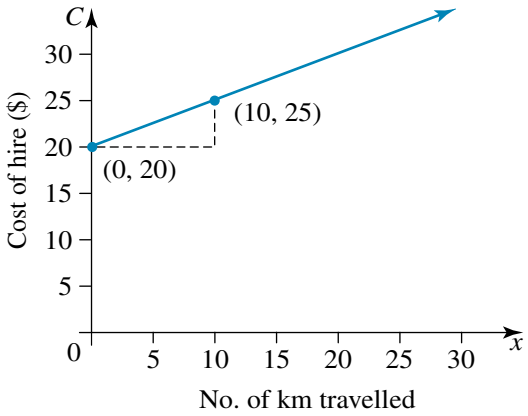
Intervals of change between linearly related variables are in direct proportion. This means there is a steady, or constant, rate of change of the dependent variable with respect to the independent variable.

1.4.1 Steady rates of change and linear rules

If the cost of hiring a bicycle is \$20 plus an additional \$0.50 per kilometre of travel then the cost increases steadily from \$20 by 50 cents for every kilometre travelled. The rate of change of the cost is \$0.50 per kilometre. Thus, if the bicycle is hired for a journey of 10 km then the cost would be $\$(20 + 10 \times 0.50)$ which equals \$25.

The relationship between the cost and the distance variables could be illustrated by plotting the ordered pairs (0, 20) and (10, 25) and joining these points with a straight line which extends beyond the point (10, 25).

Number of km travelled	0	10
Cost in dollars	20	25



The rate of increase of the cost remains constant between any two points on the line.

$$\text{For this graph, the rate of change of the cost} = \frac{\text{change in cost}}{\text{change in km travelled}}$$

$$= \frac{25 - 20}{10 - 0}$$

$$= 0.50$$

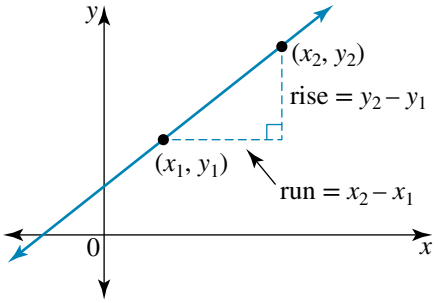
The linear rule for the hire cost, C dollars, is $C = 20 + 0.50x$, where x is the number of kilometres travelled. The rate of change of the cost appears in this rule as the coefficient of the independent variable x.

1.4.2 Gradient of a line

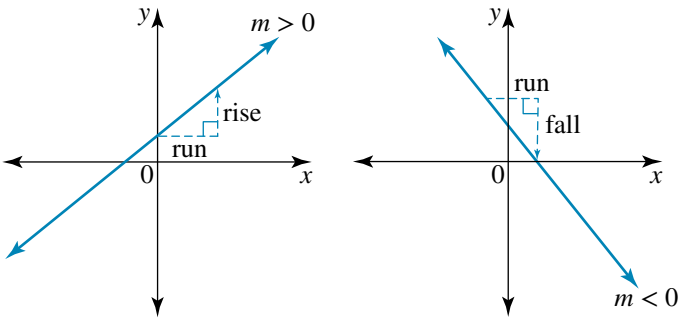
The **gradient** or **slope** measures the rate of change of the dependent variable with respect to the independent variable. Along a line, there is a steady rate of change so the gradient is constant. It measures the steepness of the line as the ratio $\frac{\text{rise}}{\text{run}}$.

The gradient of a line, usually denoted by the pronumeral m, can be calculated by choosing any two points that are known to lie on the line. If a line contains the two points with coordinates (x_1, y_1) and (x_2, y_2) then the gradient of the line is

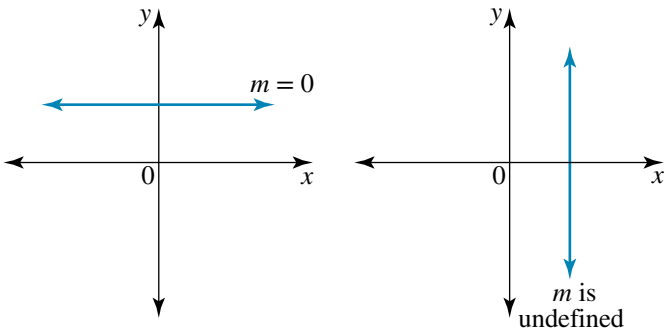
$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$



An oblique line has either a positive or a negative gradient, depending on whether the relationship between x and y is **increasing** or **decreasing**. If it is increasing, as the run increases, the rise increases so $y_2 - y_1 > 0$ and $m > 0$; if the relationship is decreasing, as the run increases, the rise decreases or falls so $y_2 - y_1 < 0$ and $m < 0$.



Horizontal lines have a gradient of zero since the rise between any two points is zero; and the gradient of a vertical line is undefined since the run between any two points is zero.



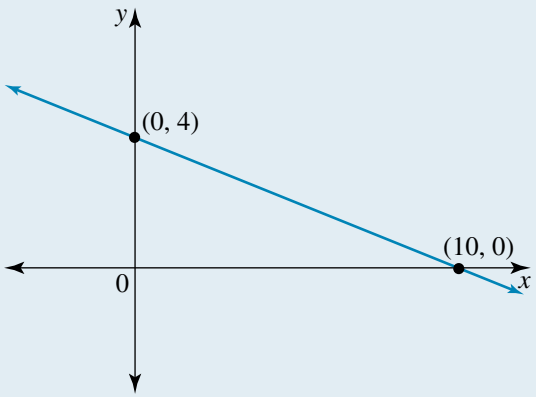
All vertical lines are **parallel**, and all horizontal lines are parallel. For other lines, if they have the same gradient then they have the same steepness, so they must be parallel to each other.

The gradient appears in the linear rule which connects the two variables x and y.

If the rule or equation is expressed in the form $y = ax + b$ then the gradient is the coefficient of x in this equation. Hence, for the line described by $y = 20 + 0.50x$ (or $y = 0.50x + 20$), the gradient is 0.50.

WORKED EXAMPLE 8

Calculate the gradient of the given line.



THINK	WRITE
1. Examine the diagram to locate two known points on the line and state their coordinates.	The intercepts with the coordinate axes are shown. Given points: $(0, 4), (10, 0)$
2. Apply the gradient formula using one point as (x_1, y_1) and the other as (x_2, y_2).	Let $(x_1, y_1) = (0, 4)$ and $(x_2, y_2) = (10, 0)$ $m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{0 - 4}{10 - 0}$ $= -\frac{4}{10}$ $\therefore m = -0.4$
3. State the answer.	The gradient of the given line is -0.4.
4. An alternative method would be to calculate the rise and the run from the diagram.	Run $= 10$, and rise (fall) $= -4$ $m = \frac{\text{rise}}{\text{run}}$ $\therefore m = \frac{-4}{10} = -0.4$

1.4.3 Sketching lines and half planes from their linear rules

While many straight lines may be drawn through a single point, only one line can be drawn through two fixed points. Two points are said to determine the line. The coordinates of every point on a line must satisfy the rule or equation which describes the line.

Sketching oblique lines

While any two points whose coordinates satisfy the equation of a line can be used to sketch it, the two points which are usually chosen are the x- and y-intercepts. Should the line pass through the origin then the coordinates of the x- and y-intercepts are both $(0, 0)$, which gives one point to use. A second point would then need to be identified. The coordinates of a second point may be obtained by substituting a value for x or y into the rule or equation that describes the line, or alternatively the gradient could be used.

Sketching horizontal and vertical lines

The y- or x-axis intercept is used to sketch a horizontal or vertical line respectively. These intercepts are evident from their equations.

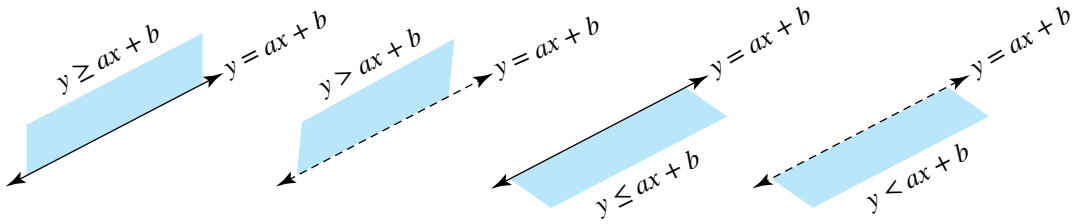
The equation $y = c$, where c is a constant, represents a horizontal line with a y-axis intercept at $(0, c)$. The equation $x = d$, where d is a constant, represents a vertical line with x-axis intercept of $(d, 0)$.

Sketching half planes

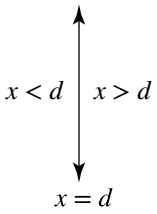
Half planes are the regions which lie on either side of a line, with the line referred to as the boundary line of the region.

If a line is described by a linear equation such as $y = ax + b$, then the linear inequations $y < ax + b$ or $y > ax + b$ or $y \leq ax + b$ or $y \geq ax + b$ describe half planes. The boundary line is included in a closed region for which $y \leq ax + b$ or $y \geq ax + b$, and not included in an open region for which $y < ax + b$ or $y > ax + b$.

Any non-vertical line divides the Cartesian plane into a region above the line and a region below the line. Since the y-coordinate measures vertical distance, the half plane, or region, above the line consists of all the points which have y-values greater than each of those of the corresponding points on the line. The half plane, or region, below the line has the points with y-values smaller than those on the line.



A vertical line also divides the Cartesian plane into two half planes, one of which consists of the points with x-coordinates smaller than that of the boundary line and the other with x-coordinates greater than that of the boundary line.



To sketch a half plane:

- sketch the boundary line, drawing it either as an open or closed line depending on the inequality sign of the region.
- shade the appropriate side of the boundary line. (We shall always shade the required region, not the unrequired region, which is an alternative convention some texts adopt.)

When sketching half planes, it may be helpful to test a point not on the boundary line to determine whether that point satisfies the inequation. If it does, then shade the side of the line containing that point; if not, shade the side of the line which does not contain the point. The origin $(0, 0)$ is usually a convenient point to use.

WORKED EXAMPLE 9

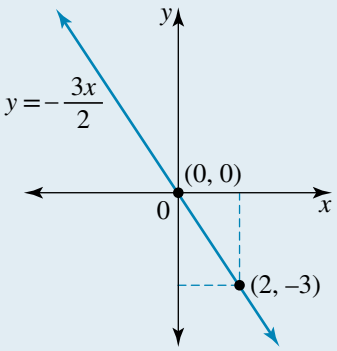
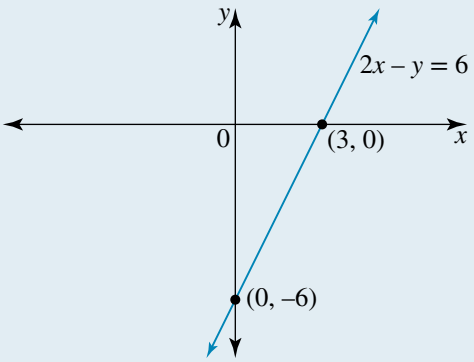
Sketch the set of points for which:

a. $y = -\frac{3x}{2}$

b. $2x - y = 6$. **Hence, sketch the region described by** $2x - y < 6$.

c. $x = 4$

d. $y \geq -2$.

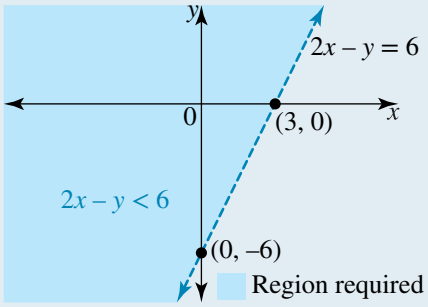
THINK	WRITE
a. 1. Calculate the y-intercept.	**a.** $y = -\frac{3x}{2}$ When $x = 0$, $y = 0$. $\Rightarrow (0, 0)$ is both the x- and the y-intercept. The line must pass through the origin.
2. A second point is needed. Substitute another value for x in the equation of the line.	Point: let $x = 2$. $y = -\frac{3 \times (2)}{2}$ $= -3$ $\Rightarrow (2, -3)$ is a point on the line.
3. Plot the two points and sketch the line.	
b. 1. Calculate the y-intercept.	**b.** $2x - y = 6$ y-intercept: let $x = 0$, $2 \times 0 - y = 6$ $-y = 6$ $\therefore y = -6$ $\Rightarrow (0, -6)$ is the y-intercept.
2. Calculate the x-intercept.	x-intercept: let $y = 0$, $2x - 0 = 6$ $2x = 6$ $\therefore x = 3$ $\Rightarrow (3, 0)$ is the x-intercept.
3. Plot the two points and draw a straight line through them. Label the points and the line appropriately.	
4. Notice the inequation indicates a half plane, or region on one side of the line $2x - y = 6$ is required. Rearrange the inequation to make y the subject.	$2x - y < 6$ $-y < -2x + 6$ $\therefore y > 2x - 6$ *Note:* The inequality sign is reversed by the division by -1.

5. Determine whether the region lies above or below the boundary line.

Region: $y > 2x - 6$
Hence, the boundary line is $y = 2x - 6$.
Therefore the required region is the open half plane above the boundary line.

6. Sketch the boundary line and shade the required region.

$y = 2x - 6$ is the same as $2x - y = 6$.
From part **b**, this boundary line has axis intercepts at $(0, -6)$ and $(3, 0)$. It is drawn as an open or non-solid line because of the $>$ sign in the inequation describing the region.
The required region is shaded on the diagram.

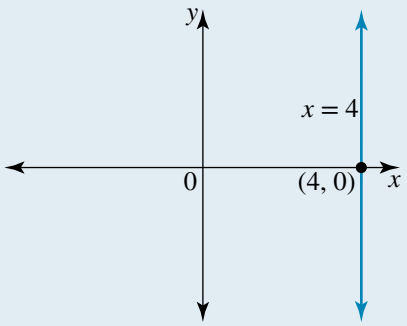


7. An alternative method is to test a point such as the origin to see if its coordinates satisfy the inequation.

Substitute $x = 0$, $y = 0$ into each side of the inequation.
LHS $= 2x - y$
$= 2(0) - (0)$
$= 0$
RHS $= 6$
Since $0 < 6$, the origin does satisfy the inequation.
Therefore the side of the line $2x - y = 6$ containing the origin should be shaded.

c. $x = 4$ is the equation of a vertical line. Sketch this line.

$x = 4$
x-intercept $(4, 0)$

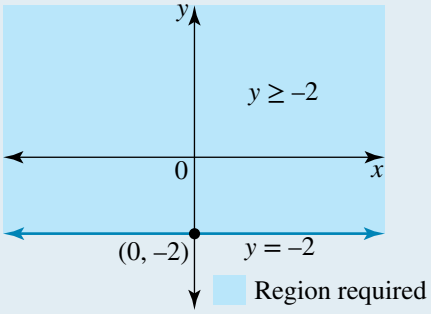


d. 1. Determine the region that is required.

d. Region: $y \geq -2$
Boundary line: $y = -2$
Therefore the required region is the closed half plane above and including the boundary line.

2. Sketch the boundary line and shade the required region.

The boundary line is a horizontal line with y-intercept $(0, -2)$.
It is drawn as a closed, or solid, line because of the $\geq$ sign in the inequation describing the region. The required region is shaded on the diagram.



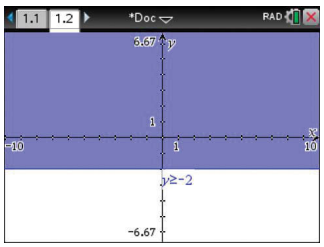
3. The alternative method of testing the origin is included as a check.

Substitute $x = 0$, $y = 0$ into each side of the inequation.
LHS $= y$
$= 0$
RHS $= -2$
Since $0 > -2$, the origin does satisfy the inequation.
Therefore, the side of the line $y = -2$ which contains the origin should be shaded.

TI | THINK

d. On a Graphs page, delete the = and select: 5. ≥
Complete the entry line as: $y \geq -2$
Then press ENTER.

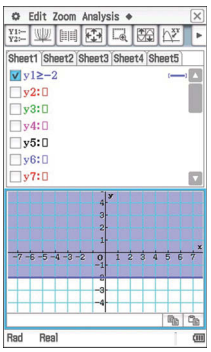
WRITE



CASIO | THINK

d. On a Graph screen, complete the entry line as: $y \geq -2$
Then tap the Graph icon.

WRITE



1.4.4 Forming equations of lines

Regardless of whether the line is oblique, horizontal or vertical, two pieces of information are required in order to form its equation. The type of information usually determines the method used to form the equation of the line. The forms of the equation of an oblique line that are most frequently used are:

- the **point–gradient form**
- the **gradient–y-intercept form.**

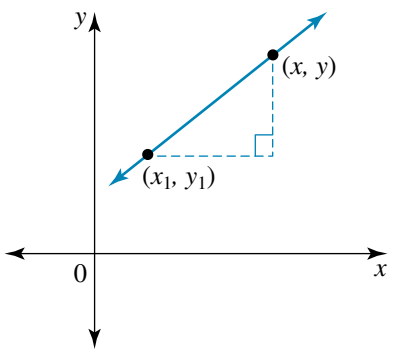
Point–gradient form of the equation of a line

Given the gradient m and a point (x_1, y_1) on the line, the equation of the line can be formed as follows:

For any point (x, y) on the line with gradient m:

$$m = \frac{y - y_1}{x - x_1}$$

$$\therefore \quad y - y_1 = m(x - x_1)$$



A line with gradient m and passing through the point (x_1, y_1) has the equation:

$$y - y_1 = m(x - x_1)$$

This equation is known as the point–gradient form of the equation of a line.

Gradient–y-intercept form

A variation on the point–gradient form is obtained if the known point is the y-intercept.

If a line with gradient m cuts the y-axis at the point $(0, c)$, then using $(0, c)$ for (x_1, y_1),

$$y - y_1 = m(x - x_1)$$
$$\therefore y - c = m(x - 0)$$
$$\therefore y = mx + c$$

The equation of a line in the form $y = mx+c$ features the gradient as the coefficient of x and the y-intercept as c. In this context it is common to refer to c as the y-intercept, with the understanding that the actual coordinates of the y-intercept are $(0, c)$.

A line with gradient m and y-intercept c has the equation:

$$y = mx + c$$

This equation is known as gradient–y-intercept form.

1.4.5 General form of the equation

The general form of the equation of a line can be written as $ax + by + c = 0$. Any equivalent form of this equation is an acceptable form for the equation of a line. For instance, the equation $3x + y - 2 = 0$ is equally as well expressed in equivalent forms which include $y = -3x + 2$, $y = 2 - 3x$, $y + 3x = 2$, $x + \frac{y}{3} = \frac{2}{3}$.

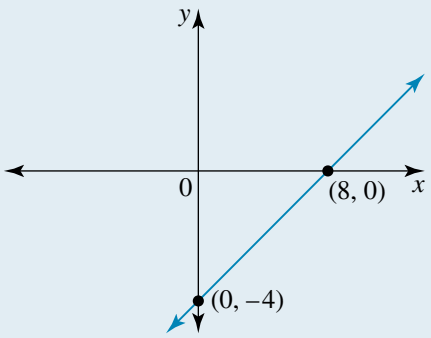
Of course, it is important to be able to recognise that the examples given are all equivalent.

WORKED EXAMPLE 10

a. Form the equation of the line with gradient 4, passing through the point $(3, -7)$.

b. Form the equation of the line passing through the points $(5, 9)$ and $(12, 0)$.

c. For the line shown, determine its equation.



d. Obtain the gradient and the coordinates of the y-intercept of the line with the equation $3x - 8y + 5 = 0$.

THINK	WRITE
a. 1. State the given information.	**a.** The gradient and a point are given. $m = 4, (x_1, y_1) = (3, -7)$
2. Write the point–gradient form of the equation.	$y - y_1 = m(x - x_1)$
3. Substitute the given information and simplify to obtain the equation.	$y - (-7) = 4(x - 3)$ $y + 7 = 4x - 12$ $\therefore y = 4x - 19$
b. 1. State the given information.	**b.** Two points are given. Let $(x_1, y_1) = (5, 9)$ and $(x_2, y_2) = (12, 0)$
2. Use the two points to calculate the gradient.	$m = \frac{0 - 9}{12 - 5}$ $= -\frac{9}{7}$
3. Write the point–gradient form of the equation.	$y - y_1 = m(x - x_1)$
4. The point–gradient equation can be used with either of the points. Substitute one of the points and simplify to obtain the equation.	Let $(12, 0)$ be the given point (x_1, y_1) in this equation. $y - 0 = -\frac{9}{7}(x - 12)$ $y = -\frac{9}{7}(x - 12)$
5. The equation could be expressed without fractions. Although this is optional, it looks more elegant.	$7y = -9(x - 12)$ $7y = -9x + 108$ $\therefore 7y + 9x = 108$
6. Had the point $(5, 9)$ been used, the same answer would have been obtained.	Check: $y - 9 = -\frac{9}{7}(x - 5)$ $7(y - 9) = -9(x - 5)$ $7y - 63 = -9x + 45$ $\therefore 7y + 9x = 108$ Same equation as before.
c. 1. Calculate the gradient from the graph (or use the coordinates of the y-intercept and the x-intercept points).	**c.** $m = \frac{\text{rise}}{\text{run}}$ $\therefore m = \frac{4}{8}$ $\therefore m = \frac{1}{2}$
2. One of the points given is the y-intercept. State m and c.	$m = \frac{1}{2}, c = -4$
3. Use the gradient–y-intercept form to obtain the required equation.	$y = mx + c$ $\therefore y = \frac{1}{2}x - 4$
d. 1. Express the equation in the form $y = mx + c$.	**d.** $3x - 8y + 5 = 0$ $\therefore 3x + 5 = 8y$ $\therefore y = \frac{3x}{8} + \frac{5}{8}$

2. State m and c. $m = \frac{3}{8}, c = \frac{5}{8}$

3. Express the answer in the required form. Gradient is $\frac{3}{8}$ and y-intercept is $\left(0, \frac{5}{8}\right)$.

on Resources

Interactivity: Equations from point–gradient and gradient–y-intercept (int-2551)

studyon

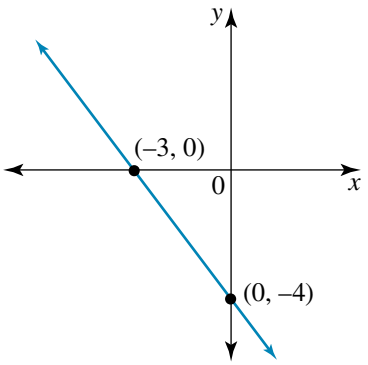
Units 1 & 2 > AOS 1 > Topic 1 > Concept 3

Linear graphs and their equations Summary screen and practice questions

Exercise 1.4 Linear graphs and their equations

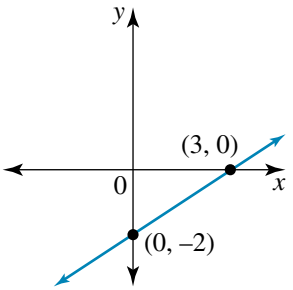
Technology free

1. WE8 Calculate the gradient of the given line.

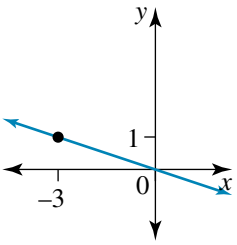


2. a. Determine the gradient of the straight line with x-intercept at $(-2, 0)$ and y-intercept at $(0, 4)$.

b. Determine the gradient of the line shown.



c. Determine the gradient of the line shown.



d. Determine the gradient of the line passing through the points $(7, -2)$ and $(2, 5)$.

3. Calculate the gradient of the line joining the following points.

a. $(-3, 8)$ and $(-7, 18)$ **b.** $(0, -4)$ and $(12, 56)$

c. $(-2, -5)$ and $(10, -5)$ **d.** $(3, -3)$ and $(3, 15)$

4. Show that the line passing through the points (a, b) and $(-b, -a)$ is parallel to the line joining the points $(-c, d)$ and $(-d, c)$.

5. WE9 Sketch the set of points for which:

a. $y = 4x$

b. $3x + 2y = 6$. Hence sketch the region described by $3x + 2y > 6$

c. $x < 3$

d. $y = 2$

6. Sketch the following linear graphs

a. $y = 3x - 6$ **b.** $y = -4x + 1$ **c.** $y = \frac{x}{2} - 5$ **d.** $y + 3x = 8$

e. $y = 5$ **f.** $y = 4x$ **g.** $y = -0.5x$ **h.** $2x = 0$

7. Sketch the lines with the following equations.

a. $y = 3x + 8$ **b.** $4y - x + 4 = 0$ **c.** $6x + 5y = 30$

d. $3y - 5x = 0$ **e.** $\frac{x}{2} - \frac{3y}{4} = 6$ **f.** $y = -\frac{6x}{7}$

8. Sketch the half planes described by the following inequations.

a. $y \leq 3 - 3x$ **b.** $4x + y > 12$

c. $5x - 2y \leq 8$ **d.** $x > -4$

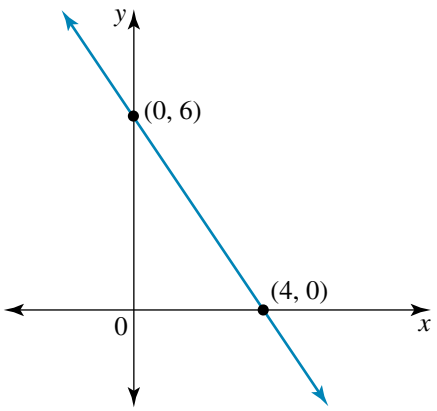
e. $y \leq 0$ **f.** $y \leq x$

9. WE10 **a.** Form the equation of the line with gradient -2, passing through the point $(-8, 3)$.

b. Form the equation of the line passing through the points $(4, -1)$ and $(-3, 1)$.

c. Determine the equation of the line shown to the right.

d. Obtain the gradient and the coordinates of the y-intercept of the line with equation $6y - 5x - 18 = 0$.



10. What is the equation of the line parallel to the x-axis that passes through the point $(2, 10)$?

11. **a.** State the equation of the line with a y-intercept at $(0, 2)$ and a gradient of 5.

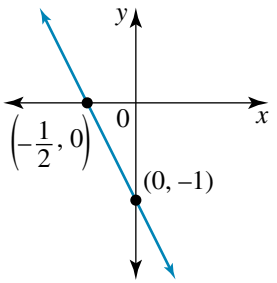
b. Determine the equation of the graph shown to the right.

c. Determine the equation of the line with a gradient of 3 and passing through the point $(1, 2)$.

d. Form the equation of the linear graph passing through the origin with a gradient of -5.

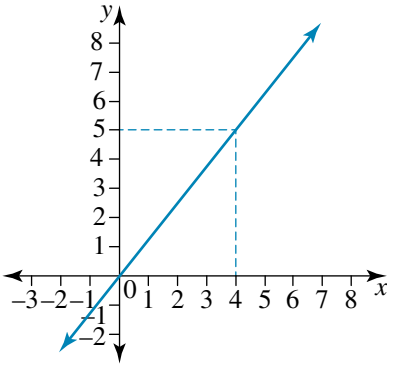
e. Form the equation of the line that passes through the points $(-1, 0)$ and $(3, -2)$.

f. Find the equation in the form $ax + by + c = 0$ for the line with gradient $-\frac{3}{4}$ and passing through the point $(-3, 5)$.



12. **a.** Form the equations of the given graphs.

i.



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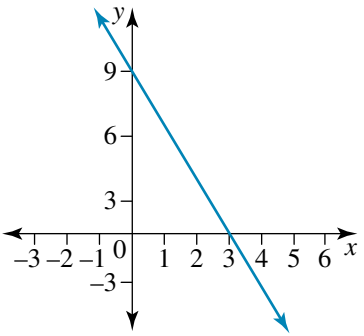
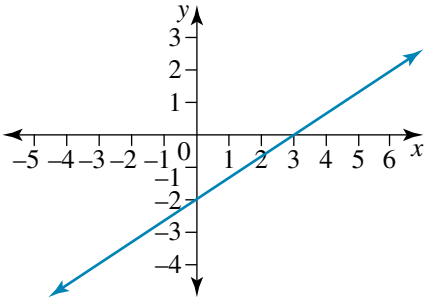
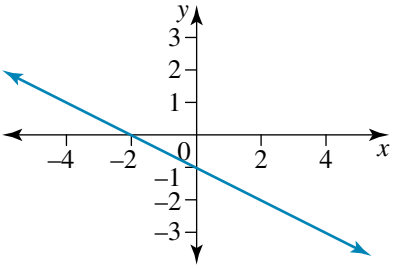


iii.



iv.



b. A line contains the points $(-12, 8)$ and $(-12, -1)$. Form its equation and sketch its graph.

c. Find the equation of the line passing through $(10, -8)$ which is parallel to the line $y = 2$, and sketch its graph.

13. Find the equations of the straight lines that are determined by the following information.

a. Gradient -5, passing through the point $(7, 2)$

b. Gradient $\frac{2}{3}$, passing through the point $(-4, -6)$

c. Gradient $-1\frac{3}{4}$, passing through the point $(0, -9)$

d. Gradient -0.8, passing through the point $(0.5, -0.2)$

e. Passing through the points $(-1, 8)$ and $(-4, -2)$

f. Passing through the points $(0, 10)$ and $(10, -10)$

14. **a.** State the gradient and the coordinates of the y-intercept of the graph which has the equation $y = 2x - 8$.

b. What is the gradient of the line with equation $5x - 3y - 6 = 0$?

c. Determine the gradient and the coordinates of the y-intercept of the line with equation $4y - 3x = 4$.

d. Which of these lines are parallel?

i. $3x - 4y - 4 = 0$

ii. $4y - 3x - 6 = 0$

iii. $6x - 8y - 6 = 0$

iv. $2y - 6x - 12 = 0$

15. Find the gradient and the y-intercept of the lines with the following equations.

a. $4x + 5y = 20$

b. $\frac{2x}{3} - \frac{y}{4} = -5$

c. $x - 6y + 9 = 0$

d. $2y - 3 = 0$

Technology active

16. Determine whether the point $(-2.2, 17.6)$ lies on, above or below the line with equation $y = 4.4 - 10x$.

17. Sketch with CAS technology the lines with the following equations.

a. $2y - 4x = -11$ **b.** $x = 5$ **c.** $y = -3$

18. Use CAS technology to sketch the regions defined by the following.

a. $2y - 4x > -11$ **b.** $x \leq 5$ and $y \geq 1$

19. a. Find the value of a if the point $(2a, 2 - a)$ lies on the line given by $5y = -3x + 4$.

b. Does the point $(-22, 13)$ lie on, above or below the line with equation $7y - 3x = 25$?

c. Form the equation of the line containing the points (p, q) and $(-p, -q)$.

20. a. If y is directly proportional to x, and $y = 52.5$ when $x = 15$:

i. find the rule for y in terms of x

ii. sketch the graph of y versus x and state its gradient.

b. If the cost, C dollars, of hiring a rowing boat is \$30 plus \$1.50 per hour, or part thereof:

i. find the rule for C in terms of the hire time in hours t

ii. sketch the graph of C versus t and state its gradient.

21. A girl's pulse rate measured 180 beats per minute immediately following an exercise session, and thereafter decreased by an amount proportional to the time that had elapsed since she finished exercising.

a. Construct a rule relating her pulse rate, p beats per minute and t, the time in minutes since she stopped the exercise.

b. If the girl's pulse rate decreases at 10 beats per minute, how long does it take for her pulse rate to return to its normal rest rate of 60 beats per minute?

c. Express p in terms of t and sketch the t vs p graph over an appropriate interval.

d. What is the gradient of the graph?

22. A family of parallel lines has the equation $3x - 2y = k$ where k is a real number.

a. What is the gradient of each member of this family of lines?

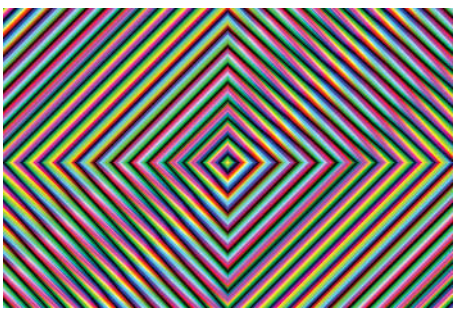
b. Show that all lines in the family contain the point (k, k).

c. Which member of the family contains the point $(-3, -8)$?

d. Which member of the family has a y-intercept of 2?

e. Sketch the lines found in part **c** and part **d** on the same set of axes.

f. Describe the closed region between these two lines using inequations.

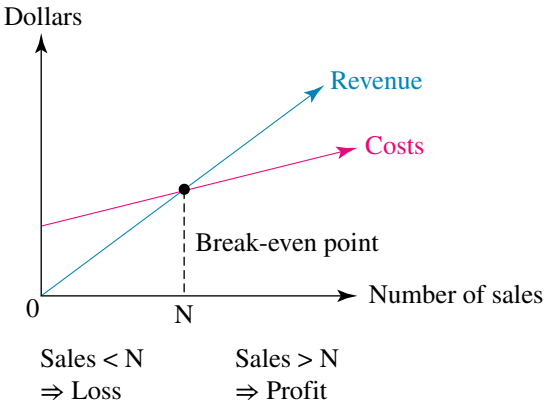


1.5 Intersections of lines and their applications

Graphs offer visual communication and allow comparisons between models to be made. If a revenue model and cost model are graphed together, it is possible to see when a profit is made.

A profit will be made only when the revenue graph lies above the cost graph. At the point of intersection of the two graphs, the break-even point, the revenue equals the costs. Before this point a loss occurs since the revenue graph lies below the cost graph.

The point of intersection of the two graphs becomes a point of interest as the number of sales needed to make a profit can be deduced from it. However, this key feature cannot often be obtained with precision by reading from the graph.



1.5.1 Intersections of lines

Two lines with different gradients will always intersect at a point. Since this point must lie on both lines, its coordinates can reliably be found algebraically using simultaneous equations.

WORKED EXAMPLE 11

The model for the revenue in dollars, d, from the sale of n items is $d_R = 20n$ and the cost of manufacture of the n items is modelled by $d_C = 500 + 5n$.

a. Find the coordinates of the point of intersection of the graphs of these two models and sketch the graphs on the same set of axes.

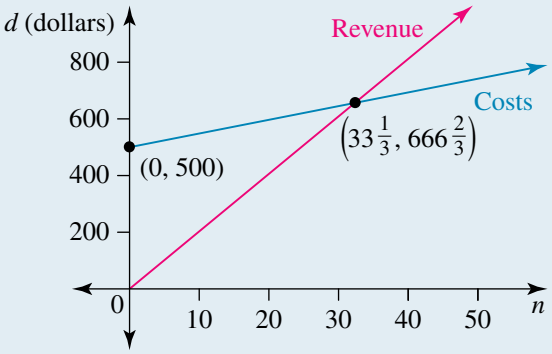
b. Obtain the smallest value of n for a profit to be made.

THINK	WRITE
a. 1. Write the equation at the point of intersection.	**a.** At the intersection, or break-even point: $d_R = d_C$ $20n = 500 + 5n$
2. Solve to find n.	$15n = 500$ $n = \frac{100}{3}$ $\therefore n = 33\frac{1}{3}$
3. Calculate the d coordinate.	When $n = \frac{100}{3}$, $d = 20 \times \frac{100}{3}$ $\therefore d = 666\frac{2}{3}$
4. State the coordinates of the point of intersection.	The point of intersection is $\left(33\frac{1}{3}, 666\frac{2}{3}\right)$.

5. Both graphs contain the intersection point. Find one other point on each graph.

Points: Let $n = 0$,
$\therefore d_R = 0$ and $d_C = 500$
The revenue graph contains the points
$(0, 0), \left(33\frac{1}{3}, 666\frac{2}{3}\right)$.
The cost graph contains the points
$(0, 500), \left(33\frac{1}{3}, 666\frac{2}{3}\right)$.

6. Sketch the graphs.



b. 1. State the condition for a profit to be made.

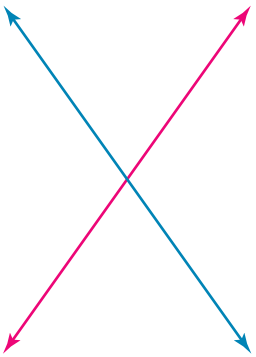
2. Answer the question.

b. For a profit, $d_R > d_C$.
From the graph $d_R > d_C$ when $n > 33\frac{1}{3}$.
Therefore, at least 34 items need to be sold for a profit to be made.

1.5.2 The number of solutions to systems of 2 × 2 linear simultaneous equations

Since a linear equation represents a straight line graph, by considering the possible intersections of two lines, three possible outcomes arise when solving a system of simultaneous equations.

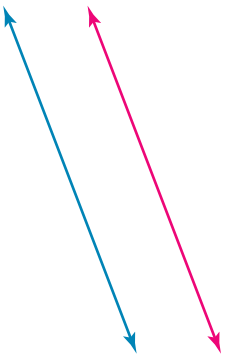
Case 1



$m_1 \neq m_2$

Case 1: Unique solution to the system. The equations represent two lines which intersect at a single point.

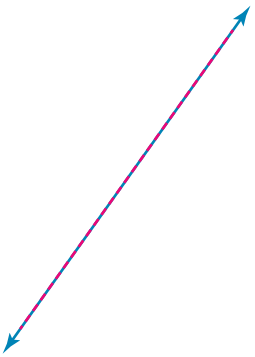
Case 2



$m_1 = m_2$
(parallel)

Case 2: No solution to the system. The equations represent parallel lines.

Case 3



$m_1 = m_2, c_1 = c_2$
(identical)

Case 3: Infinitely many solutions. The equations are equivalent and represent the same line. Every point on the line is a solution.

If the system of equations is rearranged to be of the form

$$y = m_1x + c_1$$
$$y = m_2x + c_2$$

then:

- **unique solution if $m_1 \neq m_2$**
- **no solution if $m_1 = m_2$ and $c_1 \neq c_2$**
- **infinitely many solutions if $m_1 = m_2$ and $c_1 = c_2$.**

WORKED EXAMPLE 12

Find the value of m so that the system of equations shown below has no solutions.

$$mx - y = 2$$
$$3x + 4y = 12$$

THINK	WRITE
1. Rearrange both equations to the $y = mx + c$ form.	$mx - y = 2 \Rightarrow y = mx - 2$ and $3x + 4y = 12 \Rightarrow y = -\frac{3}{4}x + 3$
2. State the gradients of the lines the equations represent.	Gradients are m and $-\frac{3}{4}$.
3. State the condition for the system of equations to have no solution, and calculate m.	For the system of equations to have no solution, the lines have to be parallel, but have different y-intercepts. For the lines to be parallel, the two gradients have to be equal. $\therefore m = -\frac{3}{4}$
4. The possibility of the equations being equivalent has to be checked.	Substitute $m = -\frac{3}{4}$ into the $y = mx + c$ forms of the equations. $y = -\frac{3}{4}x - 2$ and $y = -\frac{3}{4}x + 3$ represent parallel lines since they have the same gradients and different y-intercepts.
5. State the answer.	Therefore if $m = -\frac{3}{4}$, the system will have no solution.

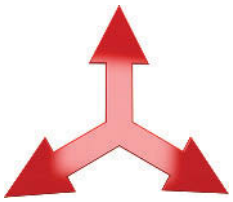
Resources

Interactivity: Intersecting, parallel and identical lines (int-2552)

1.5.3 Concurrent lines

Three or more lines that intersect at a common point are said to be **concurrent**. Their point of intersection is known as the point of concurrency.

To show that three lines are concurrent, the simplest method is to find the point of intersection of two of the lines and then check whether that point lies on the third line.



WORKED EXAMPLE 13

Show that the three lines with equations $5x + 3y = 1$, $4x + 7y = 10$ and $2x - y = -4$ are concurrent.

THINK	WRITE
1. Select the pair of equations to solve simultaneously.	Consider $4x + 7y = 10$ [1] $2x - y = -4$ [2]
2. Solve this system of equations.	Eliminate x. Multiply equation [2] by 2 and subtract it from equation [1]. $7y - 2 \times (-y) = 10 - 2 \times (-4)$ $7y + 2y = 10 + 8$ $9y = 18$ $\therefore y = 2$ Substitute $y = 2$ into equation [2]. $2x - 2 = -4$ $2x = -2$ $\therefore x = -1$ Lines [1] and [2] intersect at $(-1, 2)$.
3. Test whether the values for x and y satisfy the third equation.	Substitute $x = -1, y = 2$ into $5x + 3y = 1$. LHS $= 5 \times (-1) + 3 \times (2)$ $= 1$ $=$ RHS Therefore $(-1, 2)$ lies on $5x + 3y = 1$.
4. Write a conclusion.	Since $(-1, 2)$ lies on all three lines, the three lines are concurrent. The point $(-1, 2)$ is their point of concurrency.

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Units 1 & 2 › AOS 1 › Topic 1 › Concept 5

Coordinate geometry graphical representation Summary screen and practice questions

Exercise 1.5 Intersections of lines and their applications

Technology free

1. Use simultaneous equations to find the coordinates of the point of intersection of the lines with equations $3x - 2y = 15$ and $x + 4y = 54$.
2. Find the coordinates of the point of intersection of each of the following pairs of lines.
   a. $4x - 3y = 13$ and $2y - 6x = -7$
   b. $y = \frac{3x}{4} - 9$ and $x + 5y + 7 = 0$
   c. $y = -5$ and $x = 7$
3. A triangle is bounded by the lines with equations $x = 3$, $y = 6$ and $y = -3x$.
   a. Find the coordinates of its vertices.
   b. Calculate its area in square units.
4. **WE11** If the model for the revenue in dollars, d, from the sale of n items is $d_R = 25n$ and the cost of manufacture of the n items is modelled by $d_C = 260 + 12n$:
   a. find the coordinates of the point of intersection of the graphs of these two models and sketch the graphs on the same set of axes
   b. obtain the smallest value of n for a profit to be made.
5. The daily cost of hiring a bicycle from the Pedal On company is $10 plus 75 cents per kilometre whereas from the Bikes R Gr8 company the cost is a flat rate of $20 with unlimited kilometres.
   a. State the linear equations that model the costs of hiring the bicycles from each company.
   b. On one set of axes, sketch the graphs showing the cost versus the number of kilometres ridden, for each company.
   c. After how many kilometres are the costs equal?
   d. Shay wishes to hire a bicycle for the day. How can Shay decide from which of the two companies to hire the bicycle?
6. **WE12** Find the value of m so that the following system of equations has no solutions.
$$2mx + 3y = 2m$$
$$4x + y = 5$$
7. For what values of a and b will the following system of equations have infinitely many solutions?
$$ax + y = b$$
$$3x - 2y = 4$$
8. For what value of p will the lines $2x + 3y = 23$ and $7x + py = 8$ not intersect?
9. a. Express the lines given by $px + 5y = q$ and $3x - qy = 5q$, $(q \neq 0)$, in the $y = mx + c$ form.
   b. Hence, determine the values of p and q so the system of equations
$$px + 5y = q$$
$$3x - qy = 5q$$
   will have infinitely many solutions.
   c. What relationship must exist between p and q so the lines $px + 5y = q$ and $3x - qy = 5q$ will intersect?

Technology active

10. Use the graphing facility on CAS technology to obtain the point of intersection of the pair of lines $y = \frac{17 + 9x}{5}$ and $y = 8 - \frac{3x}{2}$, to 2 decimal places.
11. The line passing through the point $(4, -8)$ with gradient -2 intersects the line with gradient 3 and y-intercept 5 at the point Q. Find the coordinates of Q.

12. **WE13** Show that the three lines with equations $2x + 3y = 0$, $x - 8y = 19$ and $9x + 5y = 17$ are concurrent.

13. Find the value of a so that the three lines defined by $x + 4y = 13$, $5x - 4y = 17$ and $-3x + ay = 5$ are concurrent.

14. **a.** Show that the following three lines are concurrent and state their point of concurrency:
$3x - y + 3 = 0$, $5x + 2y + 16 = 0$ and $9x - 5y + 3 = 0$

b. Determine the values of d so that the three lines $x + 4y = 9$, $3x - 2y = -1$ and $4x + 3y = d$ are not concurrent.

15. A tram starting from rest travels for part of its journey at a speed, in km/h, which is directly proportional to its time, in minutes, of travel. After 2 minutes the speed of the tram is 10 km/h.

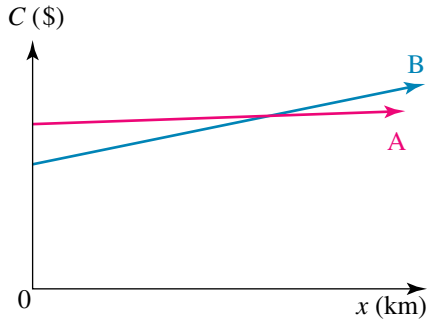
Travelling along the same route over the same period of time, the speed of a bus is partly constant and partly varies as the time of travel. Two minutes after the tram starts, the bus is travelling at 18 km/h; three minutes after that its speed has increased to 27 km/h.

a. Assuming neither the tram nor the bus stop for passengers during this time interval, construct the linear models for the speed of the tram and the speed of the bus over this time period.

b. Sketch each model on the same set of axes over a 5-minute time period.

c. How long does it take for the tram's speed to be faster than that of the bus?

16. The graph shows cost, C, in dollars, versus distance x, in kilometres, for two different car rental companies A and B.



The cost models for each company are $C = 300 + 0.05x$ and $C = 250 + 0.25x$.

a. Match each cost model to each company.

b. Explain what the gradient of each graph represents.

c. Construct a linear rule in terms of x for $y = C_A - C_B$, the difference in cost between Company A and Company B.

d. Sketch the graph of $y = C_A - C_B$ showing the intercepts with the coordinate axes.

e. Use the graph in part **d** to determine the number of kilometres when:

i. the costs of each company are the same

ii. the costs of Company A are cheaper than those of Company B.

17. The position of a boat at sea is measured as x km east and y km north of a lookout taken to be the origin $(0, 0)$. Initially, at 6 am, the boat is 2 km due north of the lookout and after 1 hour, its position is 6 km east and 3 km north of the lookout.

a. Write down the coordinates of the two positions of the boat and, assuming the boat travels in a straight line, form the equation of its path.

b. The boat continues to sail on this linear path and at some time t hours after 6 am, its distance east of the lookout is $6t$ km. At that time, show that its position north of the lookout is $(t + 2)$ km.

c. Determine the coordinates of the position of the boat at 9.30 am.

d. The positions east and north of the lighthouse of a second boat, a large fishing trawler, sailing along a linear path are given by $x = \frac{4t - 1}{3}$ and $y = t$ respectively, where t is the time in hours since 6 am. Find the coordinates of the positions of the trawler at 6 am and 7 am and hence (or otherwise) find the Cartesian equation of its linear path.

e. Show that the paths of the boat and the trawler contain a common point and give the coordinates of this point.

f. Sketch the paths of the boats on the same axes and explain whether the boat and the trawler collide.

18. At time t a particle P_1 moving on a straight line has coordinates given by $x = t, y = 3 + 2t$, while at the same time a second particle P_2 moving along another straight line has coordinates given by $x = t + 1, y = 4t - 1$.

a. Use CAS technology to sketch their paths simultaneously and so determine whether the particles collide.

b. What are the coordinates of the common point on the paths?

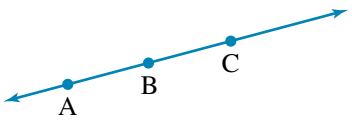
1.6 Coordinate geometry of the straight line

The gradient of a line has several applications. It determines if lines are parallel or perpendicular, it determines the angle a line makes with the horizontal and it determines if three or more points lie on the same line.

1.6.1 Collinearity

Three or more points that lie on the same line are said to be **collinear**.

If $m_{AB} = m_{BC}$ then the line through the points A and B is parallel to the line through the points B and C; but, since the point B is common to both AB and BC, the three points A, B and C must lie on the same line.



Alternatively, the equation of the line through two of the points can be used to test whether the third point also lies on that line.

WORKED EXAMPLE 14

Show that the points A(−5, −3), B(−1, 7) and C(1, 12) are collinear.

THINK	WRITE
1. Select two of the points.	For the points A and B, let (x_1, y_1) be $(-5, -3)$ and (x_2, y_2) be $(-1, 7)$.
2. Calculate the gradient of AB.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $= \frac{7 - (-3)}{-1 - (-5)}$ $= \frac{10}{4}$ $\therefore m_{AB} = \frac{5}{2}$
3. Select another pair of points containing a common point with the interval AB.	For the points B and C, let (x_1, y_1) be $(-1, 7)$ and (x_2, y_2) be $(1, 12)$.
4. Calculate the gradient of BC. *Note:* The interval AC could equally as well have been chosen.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{12 - 7}{1 - (-1)}$ $\therefore m_{BC} = \frac{5}{2}$
5. Compare the gradients to determine collinearity.	Since $m_{AB} = m_{BC}$ and the point B is common, the three points lie on the same line so they are collinear.

1.6.2 Angle of inclination of a line to the horizontal

If θ is the angle a line makes with the positive direction of the x-axis (or other horizontal line), and m is the gradient of the line, then a relationship between the gradient and this angle can be formed using trigonometry.

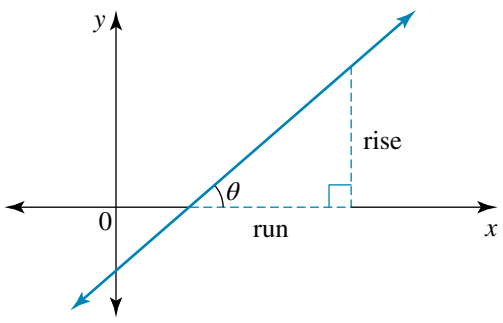
In the right-angled triangle,

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{\text{rise}}{\text{run}}$$
$$\therefore \tan\theta = m$$

Hence, the gradient of the line is given by

$$\boldsymbol{m = \tan\theta}$$

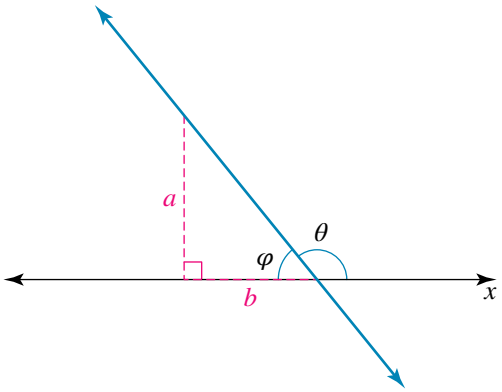
If the line is vertical, $\theta = 90°$. If the line is horizontal, $\theta = 0°$.



For oblique lines, the angle is either acute ($0° < \theta < 90°$) or obtuse ($90° < \theta < 180°$) and a calculator is used to find θ.

In the obtuse-angle case, $\theta = 180° - \varphi$ where φ is acute. If the gradient of the line is $m = -\frac{a}{b}$ then, using trigonometry, $\tan\varphi = \frac{a}{b}$ and therefore $\theta = 180° - \tan^{-1}\left(\frac{a}{b}\right)$.

Note that $\frac{a}{b}$ is the positive part of m. The mathematical term for this positive part is the **absolute value** and it can be written as $|m| = \frac{a}{b}$, or alternatively, just remember to use the positive part of the gradient when calculating the obtuse **angle of inclination**. For example, if $m = -3$, then $|m| = 3$ and the obtuse angle is calculated from $\theta = 180° - \tan^{-1}(3)$.



If $m > 0$ then θ is an acute angle, so $\theta = \tan^{-1}(m)$
If $m < 0$ then θ is an obtuse angle, so $\theta = 180° - \tan^{-1}(|m|)$

WORKED EXAMPLE 15

a. Calculate, correct to 2 decimal places, the angle made with the positive direction of the x-axis by the line which passes through the points $(-1, 2)$ and $(2, 8)$.

b. Calculate the angle of inclination with the horizontal made by a line which has a gradient of -0.6.

c. Obtain the equation of the line which passes through the point $(5, 3)$ at an angle of $45°$ to the horizontal.

THINK	WRITE
a. 1. Calculate the gradient of the line through the given points.	a. Points $(-1, 2)$ and $(2, 8)$ $m = \frac{8-2}{2-(-1)}$ $= \frac{6}{3}$ $= 2$
2. Write down the relationship between the angle and the gradient.	$\tan\theta = m$ $\therefore \tan\theta = 2$
3. Find θ correct to 2 decimal places using a calculator.	$\theta = \tan^{-1}(2)$ $\therefore \theta \simeq 63.43°$
4. Answer in context.	Therefore the required angle is $63.43°$.
b. 1. State the relationship between the angle and the gradient given.	b. The gradient -0.6 is given. $\tan\theta = m$ $\therefore \tan\theta = -0.6$
2. Write down the method for finding the obtuse angle.	The required angle is obtuse since the gradient is negative. $\theta = 180° - \tan^{-1}(\lvert m\rvert)$ Since $m = -0.6$, $\lvert m\rvert = 0.6$ $\theta = 180° - \tan^{-1}(0.6)$

3. Calculate the angle.	$\therefore \theta \simeq 149.04°$
4. Answer in context.	Therefore the required angle is 149°, to the nearest degree.
c. 1. State the relationship between the gradient and the angle.	**c.** $m = \tan\theta$
2. Substitute the given angle for θ and calculate the gradient.	$m = \tan(45°)$ $\therefore m = 1$
3. Use the point–gradient form to obtain the equation of the line.	$y - y_1 = m(x - x_1),\ m = 1,\ (x_1, y_1) = (5, 3)$ $y - 3 = 1(x - 5)$ $\therefore y = x - 2$
4. State the answer.	The equation of the line is $y = x - 2$.

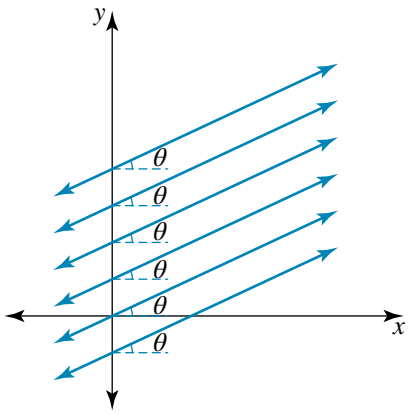
1.6.3 Parallel lines

Parallel lines have the same gradient but different y-intercepts. A set of parallel lines is called a family of lines since each line has a common feature, namely the same gradient.

If two lines with gradients m_1 and m_2 are parallel, then $m_1 = m_2$.

Each line would be inclined at the same angle to the horizontal.

A family of parallel lines



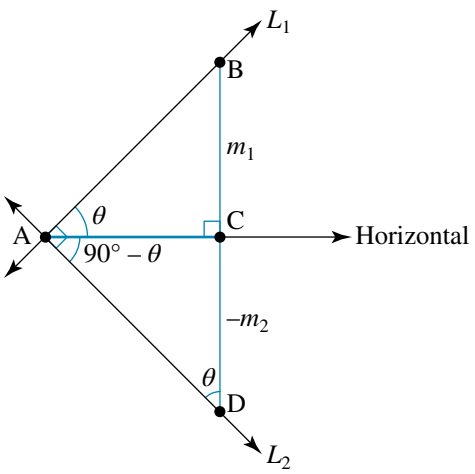
1.6.4 Perpendicular lines

A pair of lines are perpendicular to each other when the angle between them is 90°. For a pair of oblique lines, one must have a positive gradient and the other a negative gradient.

To find the relationship between these gradients consider two **perpendicular lines** L_1 and L_2 with gradients m_1 and m_2 respectively. Suppose $m_1 > 0$, and $m_2 < 0$ and that L_1 is inclined at an angle θ to the horizontal.

The diagram shows the line L_1 passing through the points A and B and the line L_2 passing through the points A and D with the angle BAD a right angle.

Taking AC as 1 unit, the sides in the diagram are labelled with their lengths. The side CB has length m_1. Since lengths must be positive, the side CD is labelled as $-m_2$ since $m_2 < 0$.



From the triangle ABC in the diagram, $\tan\theta = \frac{m_1}{1} = m_1$ and from the triangle ACD in the diagram, $\tan\theta = \frac{1}{-m_2}$.

Hence,

$$m_1 = \frac{1}{-m_2}$$

$$\therefore \quad m_1 m_2 = -1$$

This is the relationship between the gradients of perpendicular lines.

$$\boldsymbol{m_1 m_2 = -1 \quad \text{or} \quad m_2 = -\frac{1}{m_1}}$$

- If two lines with gradients m_1 and m_2 are perpendicular, then the product of their gradients is -1. One gradient is the negative reciprocal of the other.
- It follows that if $m_1 m_2 = -1$ then the two lines are perpendicular. This can be used to test for perpendicularity.

WORKED EXAMPLE 16

a. State the gradient of a line which is
i. parallel to $2y - 5x = 4$
ii. perpendicular to $2y - 5x = 4$.

b. Show that the lines $y = 4x$ and $y = -0.25x$ are perpendicular.

c. Determine the equation of the line through the point (1, 1) perpendicular to the line $y = -3x - 9$.

THINK	WRITE
a. i. 1. Rearrange the equation of the given line to express it in the $y = mx + c$ form.	**a. i.** $2y - 5x = 4$ $2y = 5x + 4$ $y = \frac{5x}{2} + \frac{4}{2}$ $\therefore y = \frac{5x}{2} + 2$
2. State the gradient of the given line.	The given line has $m = \frac{5}{2}$.
3. State the gradient of a line parallel to the given line.	The gradient of a parallel line will be the same as that of the given line. Therefore a line parallel to $2y - 5x = 4$ has a gradient of $\frac{5}{2}$.
ii. 1. State the gradient of a perpendicular line to the given line.	**ii.** The gradient of a perpendicular line will be the negative reciprocal of the gradient of the given line. If $m_1 = \frac{5}{2}$ then $m_2 = -\frac{2}{5}$. Therefore a line perpendicular to $2y - 5x = 4$ has a gradient of $-\frac{2}{5}$.

b. **1.** Write down the gradients of each line.

2. Test the product of the gradients.

b. Lines: $y = 4x$ and $y = -0.25x$

Gradients: $m_1 = 4, m_2 = -0.25$

$m_1 m_2 = 4 \times -0.25$

$\therefore m_1 m_2 = -1$

Therefore the lines are perpendicular.

c. **1.** State the gradient of the given line and calculate the gradient of the perpendicular line.

c. For $y = -3x - 9$, its gradient is $m_1 = -3$.

The perpendicular line has gradient $m_2 = -\frac{1}{m_1}$.

Therefore the perpendicular line has gradient $\frac{1}{3}$.

2. Use the point–gradient form to obtain the equation of the required line.

$m = \frac{1}{3}$ and $(x_1, y_1) = (1, 1)$

$y - y_1 = m(x - x_1)$

$\therefore y - 1 = \frac{1}{3}(x - 1)$

$\therefore 3(y - 1) = 1(x - 1)$

$\therefore 3y - 3 = x - 1$

$\therefore 3y - x = 2$

3. State the answer.

The required line has equation $3y - x = 2$.

studyon

Units 1 & 2 › AOS 1 › Topic 1 › Concept 6

Coordinate geometry of the straight line Summary screen and practice questions

Exercise 1.6 Coordinate geometry of the straight line

Technology free

1. **WE14** Show that the points A(−3, −12), B(0, 3) and C(4, 23) are collinear.
2. Determine whether the points A(−4, 13), B(7, −9) and C(12, −19) are collinear.
3. Find the value of b if the three points $(3, b)$, $(4, 2b)$ and $(8, 5 - b)$ are collinear.
4. For the points P(−6, −8), Q(6, 4) and R(−32, 34), find the equation of the line through P and Q and hence determine if the three points are collinear.
5. Explain whether or not the points A(−15, −95), B(12, 40) and C(20, 75) may be joined to form a triangle.
6. **WE16** **a.** State the gradient of a line which is:
   - **i.** parallel to $3y - 6x = 1$
   - **ii.** perpendicular to $3y - 6x = 1$.

   b. Show that the lines $y = x$ and $y = -x$ are perpendicular.

   c. Determine the equation of the line through the point $(1, 1)$ perpendicular to the line $y = 5x + 10$.
7. **a.** Determine the gradient of the line parallel to the line $x - 2y = 6$.

   b. Determine the gradient of the line parallel to the line $3y - 4x + 2 = 0$.

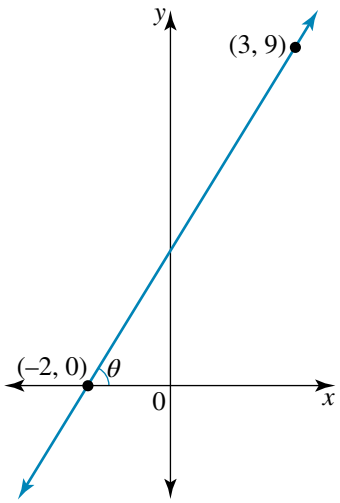
   c. Determine the gradient of the line perpendicular to $y = 3x - 4$.

   d. Determine the gradient of the line perpendicular to $4y - 2x = 8$.

8. **a.** Determine the gradient of the line perpendicular to $5x + 4y - 1 = 0$.
   b. What is the gradient of the line that is perpendicular to the line connecting the two points $(-3, 5)$ and $(2, -7)$?
   c. What is the gradient of the line that is perpendicular to the line connecting the two points $(2, 4)$ and $(7, -1)$?
   d. Show that the lines $y = 0.2x$ and $y = -5x$ are perpendicular.
9. Determine the equation of the line, in the form $ax + by = c$, which:
   a. passes through the point $(0, 6)$ and is parallel to the line $7y - 5x = 0$
   b. passes through the point $\left(-2, \frac{4}{5}\right)$ and is parallel to the line $3y + 4x = 2$
   c. passes through the point $\left(-\frac{3}{4}, 1\right)$ and is perpendicular to the line $2x - 3y + 7 = 0$
   d. passes through the point $(0, 0)$ and is perpendicular to the line $3x - y = 2$
10. Find the coordinates of the x-intercept of the line which passes through the point $(8, -2)$, and is parallel to the line $2y - 4x = 7$.

Technology active

11. **WE15** **a.** Calculate, correct to 2 decimal places, the angle made with the positive direction of the x-axis by the line which passes through the points $(1, -8)$ and $(5, -2)$.
    b. Calculate the angle of inclination with the horizontal made by a line which has a gradient of -2.
    c. Obtain the equation of the line which passes through the point $(2, 7)$ at an angle of $135°$ to the horizontal.
12. **a.** What is the gradient of the line that passes through the point $(1, 2)$ at an angle of $40°$ to the horizontal?
    b. A line passes through the x-axis inclined at an angle of $145°$ with the positive direction of the x-axis. Calculate the gradient of this line.
13. **a.** Calculate the angle of inclination with the horizontal made by the line which has a gradient of 0.5.
    b. What is the angle of inclination with the horizontal made by the line which has a gradient of -0.5?
14. Calculate the magnitude of the angle the following lines make with the positive direction of the x-axis, expressing your answer correct to 2 decimal places where appropriate.
    a.



    b. The line that cuts the x-axis at $x = 4$ and the y-axis at $y = 3$
    c. The line that is parallel to the y axis and passes through the point $(1, 5)$
    d. The line with gradient -7
15. Calculate the angle of inclination with the horizontal made by each of the lines whose gradients are 5 and 4 respectively, and hence find the magnitude of the acute angle between these two lines.
16. Determine the equation of the line which passes through the point $(-6, 12)$ making an angle of $\tan^{-1}(1.5)$ with the horizontal.

17. A line L cuts the x-axis at the point A where $x = 4$, and is inclined at an angle of 123.69° to the positive direction of the x-axis.
    a. Form the equation of the line L specifying its gradient to 1 decimal place.
    b. Form the equation of a second line, K, which passes through the same point A at right angles to the line L.
    c. What is the distance between the y-intercepts of K and L?
18. a. Find the value of a so that the line $ax - 7y = 8$ is:
        i. parallel to the line $3y + 6x = 7$
        ii. perpendicular to the line $3y + 6x = 7$.
    b. Find the value of c if the line through the points $(2c, -c)$ and $(c, -c - 2)$ makes an angle of 45° with the horizontal.
    c. Find the value of d so the line containing the points $(d + 1, d - 1)$ and $(4, 8)$ is:
        i. parallel to the line which cuts the x-axis at $x = 7$ and the y-axis at $y = -2$
        ii. parallel to the x-axis
        iii. perpendicular to the x-axis.
    d. The angle between the two lines with gradients -1.25 and 0.8 respectively has the magnitude $\alpha°$. Calculate the value of α.
19. Given the points P$(-2, -3)$, Q$(2, 5)$, R$(6, 9)$ and S$(2, 1)$, show that PQRS is a parallelogram. Is PQRS a rectangle?
20. Determine the equation of the line which passes through the point of intersection of the lines $2x - 3y = 18$ and $5x + y = 11$, and is perpendicular to the line $y = 8$.

1.7 Bisection and lengths of line segments

1.7.1 Line segments

In theory, lines are never-ending and of infinite length; however, sections of lines having two endpoints have finite lengths. These sections are called **line segments**. For simplicity, the notation AB will be used both as the name of the line segment with endpoints A and B and, in context, as the length of that line segment.

on Resources

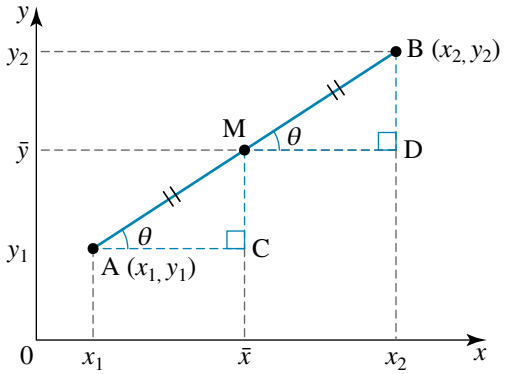
Interactivity: Midpoint of a line segment and the perpendicular bisector (int-2553)

1.7.2 The coordinates of the midpoint of a line segment

The point of bisection of a line segment is its **midpoint**. This point is **equidistant** from the endpoints of the interval.

Let the endpoints of the line segment be A(x_1, y_1) and B(x_2, y_2).

Let the midpoint of AB be the point M$(\bar{x}, \bar{y})$, where $\bar{x}$ is the mean of the x-values for A and B and $\bar{y}$ is the mean of the y-values for A and B. Since M bisects AB, AM = MB and the triangles ACM and MDB are congruent (identical).



Equating the 'runs':

$$\begin{aligned} \text{AC} &= \text{MD} \\ \bar{x} - x_1 &= x_2 - \bar{x} \\ 2\bar{x} &= x_1 + x_2 \\ \bar{x} &= \frac{x_1 + x_2}{2} \end{aligned}$$

and equating the 'rises':

$$\begin{aligned} \text{CM} &= \text{DB} \\ \bar{y} - y_1 &= y_2 - \bar{y} \\ 2\bar{y} &= y_1 + y_2 \\ \bar{y} &= \frac{y_1 + y_2}{2} \end{aligned}$$

Hence the coordinates of the midpoint of a line segment are found by averaging the coordinates of the endpoints.

The coordinates of the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2) are:

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

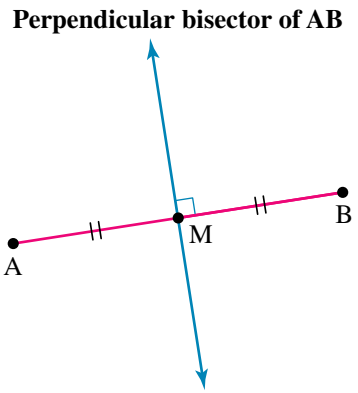
WORKED EXAMPLE 17

Calculate the coordinates of the midpoint of the line segment joining the points (−3, 5) and (7, −8).

THINK	WRITE
1. Average the x- and the y-coordinates of the endpoints.	$\bar{x} = \frac{x_1 + x_2}{2} = \frac{(-3) + 7}{2} = \frac{4}{2} = 2$ $\bar{y} = \frac{y_1 + y_2}{2} = \frac{5 + (-8)}{2} = \frac{-3}{2} = -1.5$
2. Write the coordinates as an ordered pair.	Therefore the midpoint is $(2, -1.5)$.

1.7.3 The perpendicular bisector of a line segment

The line which passes through the midpoint of a line segment and at right angles to the line segment is called the **perpendicular bisector** of the line segment.



To find the equation of the perpendicular bisector:

- its gradient is $-\frac{1}{m_{\text{AB}}}$ since it is perpendicular to the line segment AB
- the midpoint of the line segment AB also lies on the perpendicular bisector.

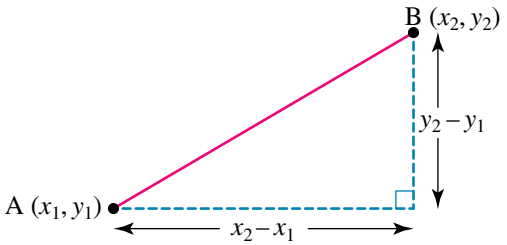
WORKED EXAMPLE 18

Determine the equation of the perpendicular bisector of the line segment joining the points A(6, 3) and B(−8, 5).

THINK	WRITE
1. Calculate the gradient of the line segment.	A(6, 3) and B(−8, 5) $m = \frac{5-3}{-8-6}$ $= \frac{2}{-14}$ $= -\frac{1}{7}$
2. Obtain the gradient of a line perpendicular to the line segment.	Since $m_{AB} = -\frac{1}{7}$, the gradient of a line perpendicular to AB is $m_{\perp} = 7$.
3. Calculate the coordinates of the midpoint of the line segment.	$\bar{x} = \frac{6+(-8)}{2}$ $\quad \bar{y} = \frac{3+5}{2}$ $= \frac{-2}{2}$ $\quad = \frac{8}{2}$ $= -1$ $\quad = 4$ Midpoint is (−1, 4).
4. Form the equation of the perpendicular bisector using the point–gradient equation.	Point (−1, 4); gradient $m = 7$ $y - y_1 = m(x - x_1)$ $y - 4 = 7(x + 1)$ $y - 4 = 7x + 7$ $\therefore y = 7x + 11$
5. State the answer.	The equation of the perpendicular bisector is $y = 7x + 11$.

1.7.4 The length of a line segment

The **length of a line segment** is the distance between its endpoints. For a line segment AB with endpoints A(x_1, y_1) and B(x_2, y_2), the run $x_2 - x_1$ measures the distance between x_1 and x_2 and the rise $y_2 - y_1$ measures the distance between y_1 and y_2.



Using Pythagoras' theorem, $(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$.

The length of the line segment AB $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

This could be expressed as the distance-between-two-points formula: $d(A,B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, where $d(A, B)$ is a symbol for the distance between the points A and B.

WORKED EXAMPLE 19

Calculate the length of the line segment joining the points A(−2, −5) and B(1, 3).

THINK	WRITE
1. Write the distance formula.	$d(\text{A}, \text{B}) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
2. Substitute the coordinates of the two points. *Note:* It does not matter which point is labelled (x_1, y_1) and which (x_2, y_2).	A(−2, −5) and B(1, 3) Let A be (x_1, y_1) and B be (x_2, y_2). $d(\text{A}, \text{B}) = \sqrt{(1 - (-2))^2 + (3 - (-5))^2}$ $= \sqrt{(3)^2 + (8)^2}$ $= \sqrt{9 + 64}$ $= \sqrt{73}$
3. State the answer. By choice, both the exact surd value and its approximate value to 2 decimal places have been given. *Note:* Always re-read the question to see if the degree of accuracy is specified.	Therefore the length of AB is $\sqrt{73} \simeq 8.54$ units.

Exercise 1.7 Bisection and lengths of line segments

Technology free

1. **WE17** Calculate the coordinates of the midpoint of the line segment joining the points (12, 5) and (−9, −1).
2. Determine the coordinates of the midpoint of the line segment joining the points
   a. (−2, 8) and (12, −2). b. (1, 0) and (−5, 4).
3. Determine the coordinates of the midpoint of the line segment joining the points
   a. (7, 3) and (−4, 2). b. (24, 12) and (16, 12)
4. **MC** M is the midpoint of the line segment AN. Given that M has coordinates (5, 6) and A is the point (3, 7), select the correct coordinates for point N.
   A. (4, 6.5) **B.** (4, −2) **C.** (7, 5) **D.** (1, 8) **E.** (−7, −5)
5. If the midpoint of PQ has coordinates (3, 0) and Q is the point (−10, 10), find the coordinates of point P.
6. Determine the equation of the line that has a gradient of −3 and passes through the midpoint of the segment joining (5, −4) and (1, 0).
7. Given the points A(3, 0), B(9, 4), C(5, 6) and D(−1, 2), show that AC and BD bisect each other.
8. **WE18** Determine the equation of the perpendicular bisector of the line segment joining the points A(−4, 4) and B(−3, 10).
9. Determine the equation of the perpendicular bisector of the line segment joining the points A(−6, 0) and B(2, 4).
10. Determine the equation of the perpendicular bisector of the line segment joining the points A(1, 2) and B(−3, 5).
11. Given that the line $ax + by = c$ is the perpendicular bisector of the line segment CD where C is the point (−2, −5) and D is the point (2, 5), find the smallest non-negative values possible for the integers a, b and c.

12. **WE19** Calculate the length of the line segment joining the points $(6, -8)$ and $(-4, -5)$.

13. Calculate the distance between the points $(10, -3)$ and $(-2, 6)$.

14. Find the exact distance between the points

a. $(-3, 2)$ and $(3, -4)$. **b.** $(-1, -5)$ and $(5, -1)$.

15. Triangle ABC has vertices A$(-2, 0)$, B$(2, 3)$ and C$(3, 0)$. Determine which of its sides has the shortest length.

Technology active

16. Calculate the distance between the point $(3, 10)$ and the midpoint of the line segment AB where A is the point $(-1, 1)$ and B is the point $(6, -1)$. Give the answer correct to 2 decimal places.

17. If the distance between the points $(p, 8)$ and $(0, -4)$ is 13 units, find two possible values for p.

18. Given the points A$(-7, 2)$ and B$(-13, 10)$, obtain:

a. the distance between the points A and B

b. the coordinates of the midpoint of the line segment AB

c. the equation of the perpendicular bisector of AB

d. the coordinates of the point where the perpendicular bisector meets the line $3x + 4y = 24$.

19. Triangle CDE has vertices C$(-8, 5)$, D$(2, 4)$ and E$(0.4, 0.8)$.

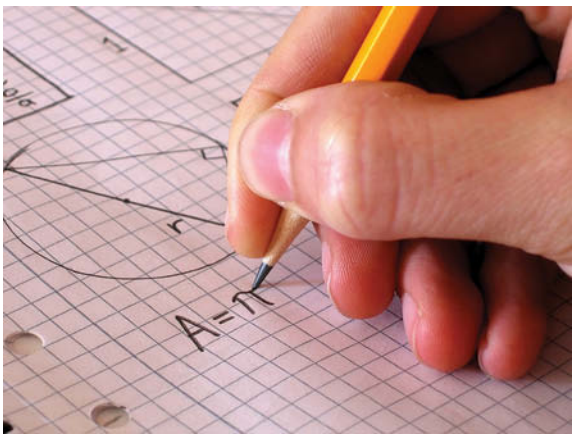
a. Calculate its perimeter to the nearest whole number.

b. Show that the magnitude of angle CED is 90°.

c. Find the coordinates of M, the midpoint of its hypotenuse.

d. Show that M is equidistant from each of the vertices of the triangle.

20. A circle has its centre at $(4, 8)$ and one end of the diameter at $(-2, -2)$.

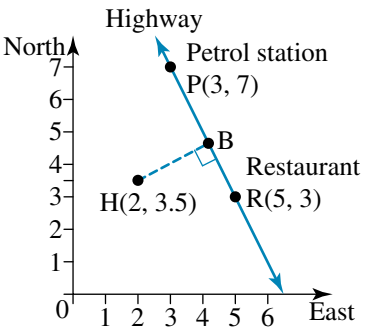


a. Specify the coordinates of the other end of the diameter.

b. Calculate the area of the circle as a multiple of π.

21. Two friends planning to spend some time bushwalking argue over which one of them should carry a rather heavy rucksack containing food and first aid items. Neither is keen so they agree to each throw a small coin towards the base of a tree and the person whose coin lands the greater distance from the tree will have to carry the rucksack. Taking the tree as the origin, and the distances in centimetres east and north of the origin as (x, y) coordinates, Anna's coin lands at $(-2.3, 1.5)$ and Liam's coin lands at $(1.7, 2.1)$. Who carries the rucksack, Anna or Liam? Support your answer with a mathematical argument.

22. The diagram shows a main highway through a country town. The section of this highway running between a petrol station at P and a restaurant at R can be considered a straight line. Relative to a fixed origin, the coordinates of the petrol station and restaurant are P(3, 7) and R(5, 3) respectively. Distances are measured in kilometres.



a. How far apart are the petrol station and restaurant? Answer to 1 decimal place.

b. Form the equation of the straight line PR.

Ada is running late for her waitressing job at the restaurant. She is still at home at the point H(2, 3.5). There is no direct route to the restaurant from her home, but there is a bicycle track that goes straight to the nearest point B on the highway from her home. Ada decides to ride her bike to point B and then to travel along the highway from B to the restaurant.

c. Form the equation of the line through H perpendicular to PR.

d. Hence find the coordinates of B, the closest point on the highway from her home.

e. If Ada's average speed is 10 km/h, how long, to the nearest minute, does it take her to reach the restaurant from her home?

For questions **23a** and **24a**, use the geometry facility on CAS technology to draw a triangle.

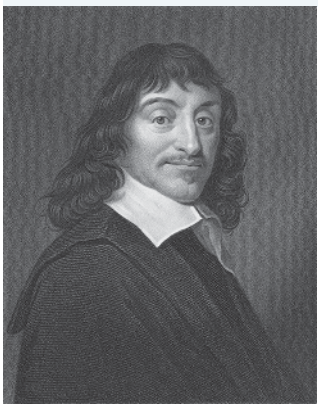
23. a. Construct the perpendicular bisectors of each of the three sides of the triangle. What do you notice? Repeat this procedure using other triangles. Does your observation appear to hold for these triangles?

b. For the triangle formed by joining the points O(0, 0), A(6, 0), B(4, 4), find the point of intersection of the perpendicular bisectors of each side. Check your answer algebraically.

24. a. Construct the line segments joining each vertex to the midpoint of the opposite side (these are called medians). What do you notice? Repeat this procedure using other triangles. Does your observation appear to hold for these triangles?

b. For the triangle formed by joining the points O(0, 0), A(6, 0) and B(4, 4), find the point of intersection of the medians drawn to each side. Check your answer algebraically.

René Descartes, seventeenth century French mathematician and philosopher, was one of the first to combine algebra and geometry together as coordinate geometry in his work *La Géométrie.*



1.8 Review: exam practice

A summary of this topic is available in the Resources section of your eBookPLUS at www.jacplus.com.au.

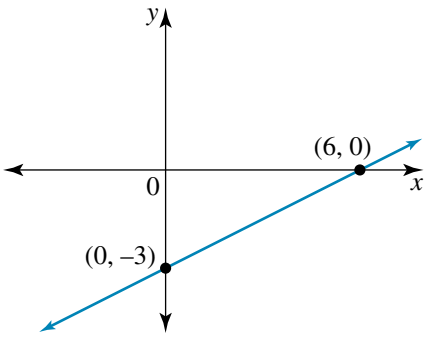
Short answer: technology free

1. Solve for x.
   a. $3(5x - 2) + 5(3x - 2) = 8(x - 2)$
   b. $\frac{2x - 1}{5} + \frac{3 - 2x}{4} = \frac{3}{20}$
   c. $ax + 3c = 3a + cx$
   d. $\frac{5x - b}{2b} - \frac{2x}{b} = 2$
2. Solve the following system of equations for x, y and z.
   $2x + y + z = 6$
   $3x + 4y - z = 4$
   $5x - 2y + z = 36$
3. Solve for x.
   a. $3 - 2x \leq 1$
   b. $\frac{2x}{3} - 5 > 2 + 3x$
4. Sketch, labelling any axes intercepts with their coordinates:
   a. the line with equation $3x - 4y = 24$
   b. the half plane defined by $3x - 2y < 0$.
5. Find the equation of the line:
   a. through the point $(-5, 8)$ parallel to the line $2x - 7y = 2$
   b. through the point $(4, 0)$ perpendicular to the line $2x - 7y = 2$
   c. through the two points $(9, 2)$ and $(6, -7)$.
6. Find the length of the line segment from the point $(-2.5, 8)$ to the midpoint of the line segment joining the points $(-4, 6)$ and $(5, -2)$.

Multiple choice: technology active

1. MC If $4(1 - 3x) + 2(3 + x) > 5$ then:
   A. $x > 2$ **B.** $x < 2$ **C.** $x < -2$ **D.** $x < \frac{1}{2}$ **E.** $x > \frac{1}{2}$
2. MC The solutions to the simultaneous equations $7x - 2y = 11$ and $3x + y = 1$ are:
   A. $x = 9, y = -26$
   B. $x = 5, y = 11$
   C. $x = -1, y = 2$
   D. $x = 1, y = -2$
   E. $x = 1, y = 4$
3. MC A calculator purchased for \$200 depreciates each year by an amount directly proportional to the number of years of ownership. If the constant of proportionality is 25, then the value V of the calculator after t years is:
   A. $V = 25t$
   B. $V = -25t$
   C. $V = 200 - 25t$
   D. $V = 200 + 25t$
   E. $V = 25t - 200$

4. MC The equation of the graph shown is:



A. $x - 2y = 6$
B. $2y + x = 6$
C. $6x - 3y = 1$
D. $y = -\frac{1}{2}x - 3$
E. $y = 2x - 3$

5. MC For the points $(-1, -2)$, $(4, 3)$ and $(9, b)$ to be collinear, b would equal:
A. -3 **B.** -2 **C.** -1 **D.** 8 **E.** 10

6. MC The angle of inclination to the positive direction of the x-axis made by the line with equation $\frac{x}{3} - \frac{y}{2} = 1$ is closest to:
A. 26.6° **B.** 33.7° **C.** 56.3° **D.** 123.7° **E.** 146.3°

7. MC The equation of the line through $(9, 5)$ parallel to $y = -2$ is:
A. $y = -2x + 13$ **B.** $2y = x + 1$ **C.** $x = 9$ **D.** $y = 9$ **E.** $y = 5$

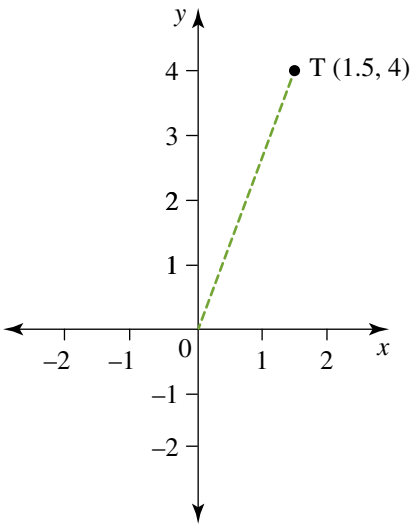
8. MC The midpoint of a line segment AB is $(3, -5)$. If A is the point $(13, 11)$, then the coordinates of B are:
A. $(8, 3)$ **B.** $(-23, -27)$ **C.** $(23, 27)$ **D.** $(-7, -21)$ **E.** $(7, 21)$

9. MC The value of a such that there would be no point of intersection between the two lines $ay + 3x = 4$ and $2y + 4x = 3$ is:
A. 2 **B.** 1.5 **C.** $\frac{8}{3}$ **D.** -0.5 **E.** -2

10. MC The distance between the points $(-3, 5)$ and $(-6, 12)$ is:
A. 4 **B.** $\sqrt{40}$ **C.** $\sqrt{58}$ **D.** $\sqrt{370}$ **E.** $\sqrt{388}$

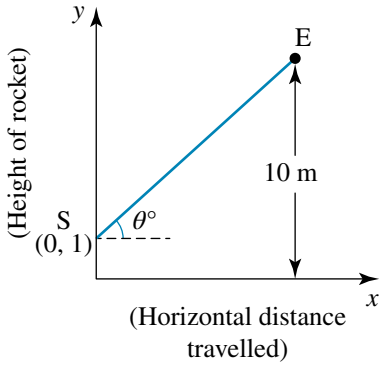
Extended response: technology active

1. After school, Tenzin rides his bike along a straight path from school to a golf range. He travels one third of the way at an average speed of 20 km/h and the remainder of the way at 10 km/h.



   a. Find the distance he travels from his school to the golf range if the journey takes 45 minutes.
   At the golf range Tenzin is trying to improve his putting. A set of Cartesian coordinates can be imagined to be marked so that Tenzin will be aiming to hit the ball at the point T$(1.5, 4)$ into the hole at the origin $(0, 0)$.
   b. If all units are measured in metres, how far is Tenzin's golf ball from the hole?
   c. What should be the equation of the straight-line path Tenzin's golf ball needs to travel along for the ball to reach the hole?
   d. In fact, Tenzin hits the golf ball along the path with equation $6x - 2y = 1$.
      i. Sketch this path.
      ii. Find the equation of the line through $(0, 0)$ perpendicular to this path and hence find, to the nearest centimetre, the closest distance of the golf ball's path to the hole at $(0, 0)$.

2. A small fireworks rocket travels along a path that can be considered to be a straight line. On a Cartesian set of x- and y-axis where the units are in metres, the x-coordinates give the horizontal distance the rocket travels and the y-coordinates give the height of the rocket above the ground.
The fireworks rocket is launched from a point S(0, 1) at an angle of $\theta°$ with the horizontal. The fireworks explode on reaching a point E, which is at a height of 10 metres above the ground.



- **a. i.** The first rocket is launched at an angle of 45° to the horizontal. Find the equation of its path and the coordinates of point E.
- **ii.** After the explosion, part of the debris travels from point E along a line perpendicular to the rocket's path. Find the equation of this path, and work out how far horizontally from E the debris reaches the ground.
- **b.** The angle at which the rockets are launched from S(0, 1) is varied and the fireworks explode at E(k, 10), $k > 0$. Show that the equation of the paths of all possible rockets is given by $9x - ky + k = 0$.
- **c. i.** Let r be the horizontal distance from E that the debris travelling on a line perpendicular to the path SE reaches the ground. Form an expression for r in terms of k.
- **ii.** It is desirable for the debris not to be too widely scattered. Find the possible values for k such that $4 \leq r \leq 6$.

3. Rain water is collected in a water tank. On 1 April the tank contained 1000 litres of water. Ten days later it contained 1250 litres. Assume the amount of water increases uniformly.

- **a.** Find the rate of increase in litres per day.
- **b.** Form the linear relationship between the volume V litres of water in the tank t days after 1 April. Which one of V or t is the independent variable?
- **c.** Describe the relationship between these variables.
- **d.** How much water does the linear model predict should have been in the tank on 30 March?

The tank needs replacing and quotes are obtained from two companies. The Latasi company charges $500 for materials plus $26 per hour of construction. The Natano company charges $600 for materials plus $18 per hour of construction.

- **e. i.** Form linear models for the costs for each company, defining symbols used.
- **ii.** When are the costs the same for each company and what is this cost?
- **iii.** Sketch both cost models on the same axes.
- **iv.** It is estimated that the construction should take approximately 8 hours of work. Which company could do the job at a cheaper cost?

4. ABCD is a quadrilateral. The coordinates of A and B are (1, 3) and (2, 1) respectively. C has the same y-coordinate as the midpoint of AB and D lies on the perpendicular bisector of AC.

- **a.** Determine the equation of BC given that BC is perpendicular to AB.
- **b.** Obtain the coordinates of point C.
- **c.** Find the equation of the perpendicular bisector of AC and show B lies on this line.
- **d.** Point E lies on BC, and C is the midpoint of BE. Obtain the coordinates of E.
- **e.** The closest point to D on the line passing through B and C is the point E. Determine the coordinates of D.
- **f.** Describe the quadrilateral ABCD.

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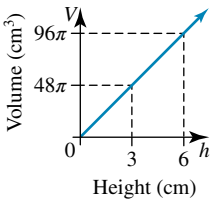
Units 1 & 2 Sit topic test

Answers

Topic 1 Lines and linear relationships

Exercise 1.2 Linearly related variables, linear equations and inequations

1. a. $k = 16\pi$ **b.** Height 3 cm



2. a. Interest is directly proportional to number of years invested, $I = 46T$
b. Not linearly related
c. Wage is a fixed amount plus an amount proportional to the number of hours of overtime; $W = 400 + 50n$

3. a. $x = -1$ **b.** $x = 10$

4. a. $x = 8$ **b.** $x = -10$ **c.** $x = -\frac{7}{3}$
d. $x = -7$ **e.** $x = \frac{31}{5}$ **f.** $x = -\frac{6}{7}$
g. $x = -\frac{46}{19}$ **h.** $x = \frac{26}{7}$

5. a. $x = 9$ **b.** $x = 20$ **c.** $x = 6$
d. $x = \frac{1}{3}$ **e.** $x = 1$ **f.** $x = -10$

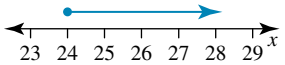
6. a. $C = 210 + 6n$ **b.** \$360

7. a. $m = 8(x - z)$ **b.** $m = \frac{a}{b - cd}$
c. $m = \frac{3d - c}{3}$ **d.** $m = \frac{d + k}{d - c}$
e. $m = \frac{2b}{bc - 3}$ **f.** $m = \frac{4d + 3a}{2a + d}$

8. $x = a + d$

9. a. $x = \frac{c - b}{a}$ **b.** $x = \frac{ab}{a - b}$ **c.** $x = -1$
d. $x = ab$ **e.** $x = \frac{a}{b - c}$ **f.** $x = b - a$

10. $x \geq 24$



11. a. $x \leq -\frac{1}{2}$ **b.** $x < -3$ **c.** $x > -2$
d. $x < -\frac{1}{3}$ **e.** $x > 0$ **f.** $x > 22$

12. a. $x \leq -1$ **b.** $x > -9$ **c.** $x \leq 18$
d. $x > 3$ **e.** $x \geq \frac{2}{3}$ **f.** $x < -6$

13. a. $x = 7, y = 1$ **b.** $x = -4, y = -2$

14. a. $x = 2, y = -1$ **b.** $x = -3, y = \frac{1}{2}$
c. $x = 2, y = 1$ **d.** $x = -1, y = 4$
e. $x = 1, y = 3$ **f.** $x = -5, y = 6$

15. a. $x = 1, y = 4$ **b.** $x = -2, y = 2$
c. $x = -\frac{1}{2}, y = -11$ **d.** $x = \frac{3}{4}, y = \frac{2}{3}$
e. $x = 12, y = 6$ **f.** $x = 3b, y = a$

16. $x = -\frac{1}{7}$ **17.** $x = c$ **18.** $x > -\frac{1}{6}$

19. a. $x = -\frac{7}{3}, y = -\frac{35}{3}$ **b.** $x = 1, y = 0$

20. a. $x = 0$ **b.** $x > 4$
c. $x = 5, y = -1$ **d.** $x = -82, y = -120$

21. a. $T = 3.5x$; tension is 1.05 newton
b. $C = 1.45l$, 30 litres
c. $v = 12 - 9.8t$; approximately 1.22 seconds

22. a. 8 and 10
b. 21
c. 12, 13 and 14
d. Length is 20 cm; width is 4 cm
e. Height is 45 cm

23. a. Profit for sale of n books is $P = 2.3n - 100$
b. 44 books
c. Hardcovers \$5; paperbacks \$2

24. a. Adult ticket costs \$25; child ticket costs \$12
b. \$471.09
c. $C = \frac{5}{9}(F - 32)$

25. 22.5 minutes

26. a. $t = 2.5$; distance is 150 km
b. $\frac{10u}{9}$

27. a. $12 \leq n \leq 16$ where the whole number n is the number of people
b. 15 people attended at a cost of \$27.90 per person.
c. 10 drank coffee, 5 drank tea

28. a. $C = 10T - 97$
b. $x = \frac{c}{a + b}, y = \frac{1}{a + b}$

Exercise 1.3 Systems of 3×3 simultaneous linear equations

1. a. i. z
ii. Subtract the equations
b. D
c. $7x + 3y = 15$
d. $5x + y = 13$
e. Substitute the given values in each equation.
f. $z = 7$

2. a. Equation [4] is $3x + y = 10$ and equation [5] is $9x + y = 34$.
b. $x = 4, y = -2$
c. $z = -5$

3. $x = 5, y = 8, z = -6$

4. $x = 1, y = -2, z = 3$

5. $x = 2, y = 1, z = 4$

6. $x = 4, y = 2, z = -1$

7. $x = -6, y = 8, z = 1$

8. $x = -2, y = 5, z = 10$

9. $x = 4, y = 3, z = 2$

10. $x = -2, y = -11, z = 40$

11. Adult ticket $14; concession ticket $12; child ticket $10

12. Agnes $20 per hour; Bjork $18 per hour; Chi $25 per hour

13. The food compound requires 50 kg of supplement X, 30 kg of supplement Y and 10 kg of supplement Z.

14. Two 50-cent coins, twelve 20-cent coins and eight 10-cent coins

15. **a.** $x = 3, y = 1.5, z = -2.6$
    b. $x = 10, y = -6, z = 0.5, w = 5$
    c. $x_1 = -2, x_2 = -4, x_3 = \frac{2}{3}, x_4 = \frac{1}{6}$

16. **a.** $z = -4 + 0.1x + 0.075y$
    b. 15 kg/hectare

Exercise 1.4 Linear graphs and their equations

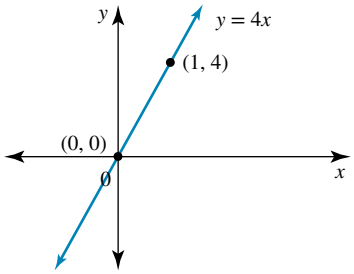
1. $m = -\frac{4}{3}$

2. **a.** 2 **b.** $\frac{2}{3}$
   c. $-\frac{1}{3}$ **d.** $-\frac{7}{5}$

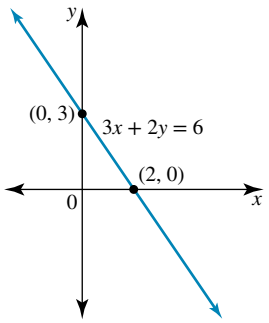
3. **a.** $m = -2.5$ **b.** $m = 5$
   c. $m = 0$ **d.** Undefined

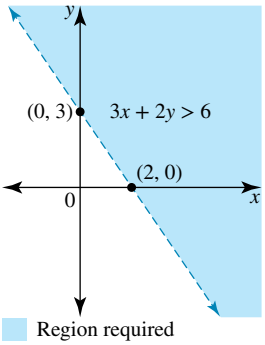
4. Show both gradients equal 1; sample responses can be found in the worked solutions in the online resources.

5. **a.**



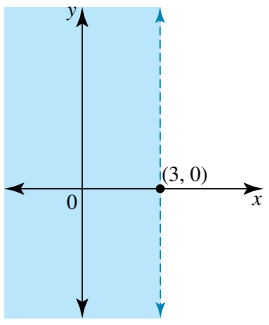
b.





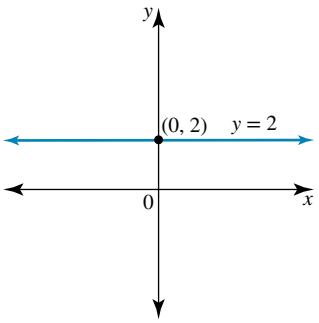
Region required

c.

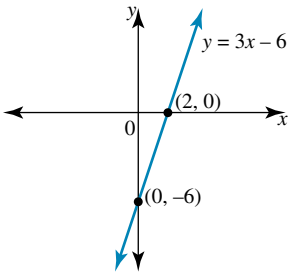


Region required

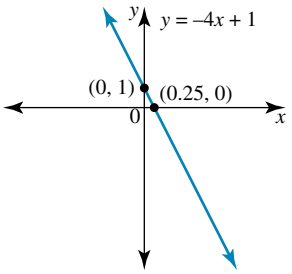
d.



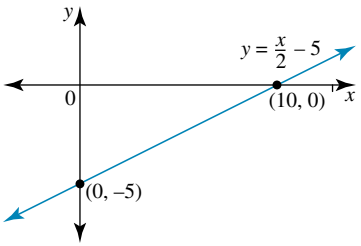
6. **a.**



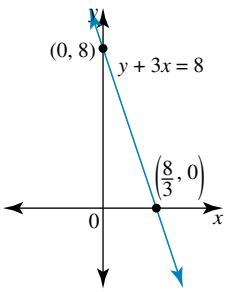
b.



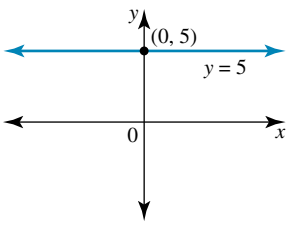
c.



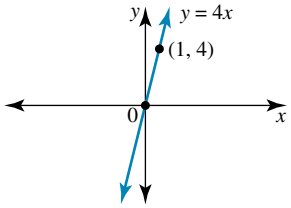
d.



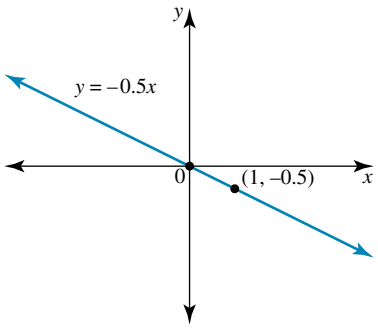
e.



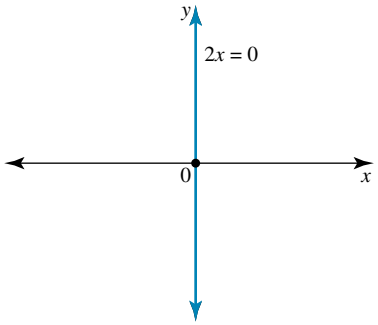
f.



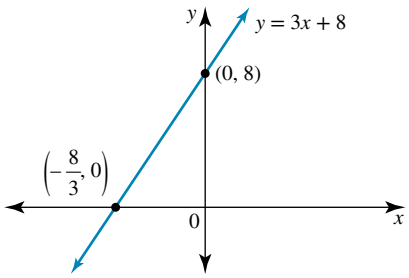
g.



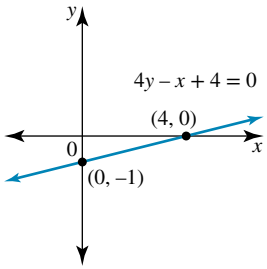
h.



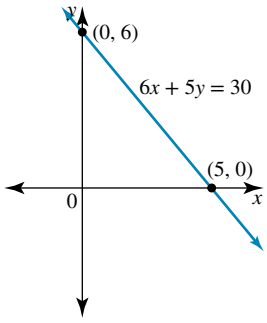
7. a.



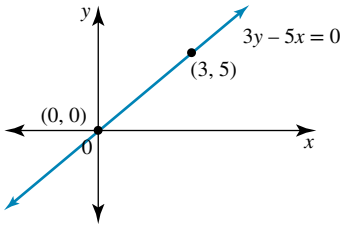
b.



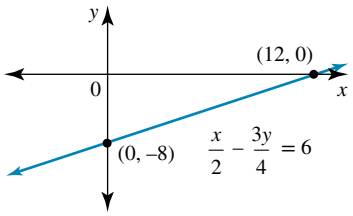
c.



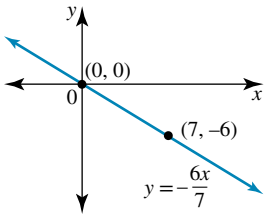
d.



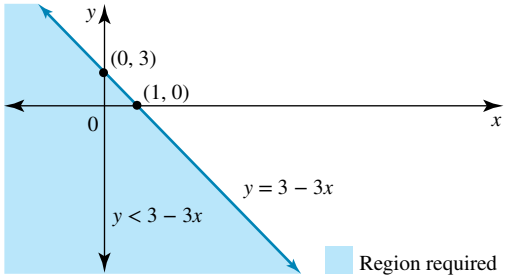
e.



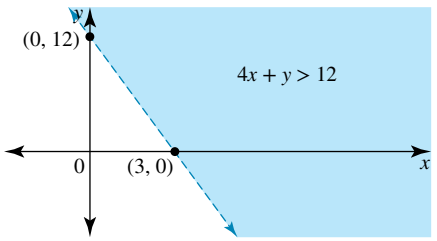
f.



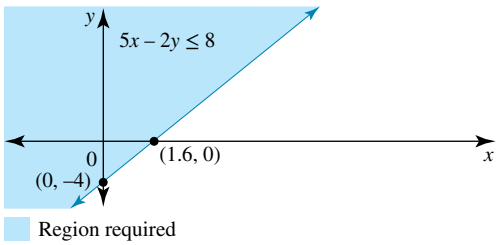
8. a. Closed region below line



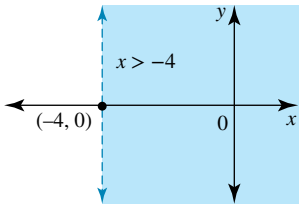
b. Open region above line



c. Closed region above line

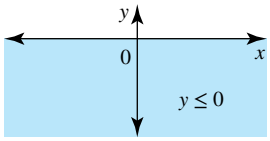


d. Open region to right of vertical line



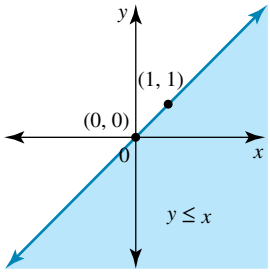
Region required

e. Closed region below x-axis



Region required

f. Closed region below line



Region required

9. a. $y = -2x - 13$ **b.** $7y + 2x = 1$

c. $y = -1.5x + 6$ **d.** $m = \frac{5}{6}$; $(0, 3)$

10. $y = 10$

11. a. $y = 5x + 2$ **b.** $y = -2x - 1$

c. $y = 3x - 1$ **d.** $y = -5x$

e. $x + 2y + 1 = 0$ **f.** $3x + 4y - 11 = 0$

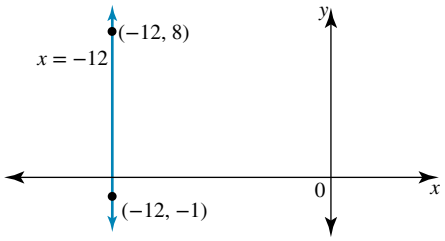
12. a. i. $y = \frac{5x}{4}$

ii. $y = -3x + 9$

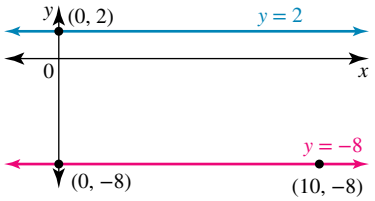
iii. $y = \frac{2}{3}x - 2$

iv. $y = -\frac{1}{2}x - 1$

b. $x = -12$



c. $y = -8$



13. a. $y = -5x + 37$ **b.** $3y - 2x + 10 = 0$

c. $4y + 7x + 36 = 0$ **d.** $y = -0.8x + 0.2$

e. $3y - 10x = 34$ **f.** $y = -2x + 10$

14. a. $m = 2$, $(0, -8)$ **b.** $\frac{5}{3}$

c. $m = \frac{3}{4}$, $(0, 1)$ **d.** **i**, **ii** and **iii** are parallel

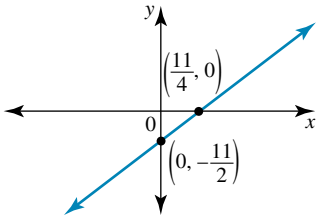
15. a. $m = -\frac{4}{5}, c = 4$ **b.** $m = \frac{8}{3}, c = 20$

c. $m = \frac{1}{6}, c = \frac{3}{2}$ **d.** $m = 0, c = \frac{3}{2}$

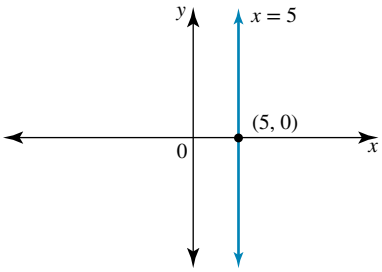
16. $y < 4.4 - 10x$

Point lies below the line

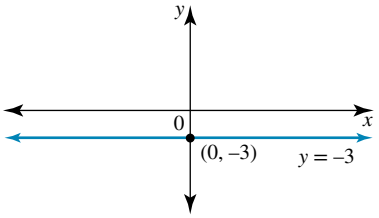
17. a.



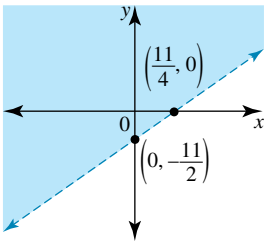
b.



c.

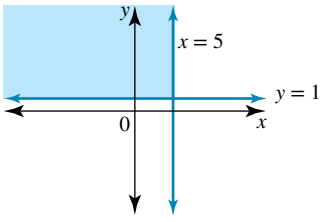


18. a.



Region required

b.



Region required

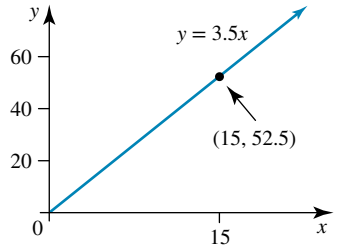
19. a. $a = -6$

b. Above the line

c. $y = \frac{q}{p}x$

20. a. i. $y = 3.5x$

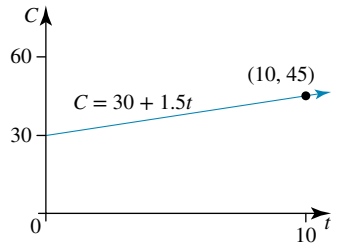
ii.



Gradient is 3.5

b. i. $C = 30 + 1.5t$

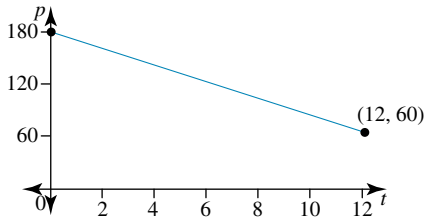
ii.



Gradient is 1.5

21. a. $p = 180 - kt$ **b.** 12 minutes

c. $p = 180 - 10t$ **d.** Gradient $= -10$



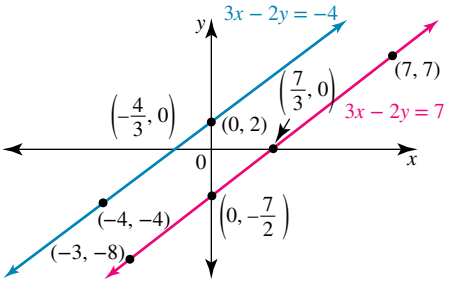
22. a. $m = \frac{3}{2}$

b. Sample responses can be found in the worked solutions in the online resources.

c. $3x - 2y = 7$

d. $3x - 2y = -4$

e.



f. $3x - 2y \leq 7$ and $3x - 2y \geq -4$, that is $-4 \leq 3x - 2y \leq 7$

Exercise 1.5 Intersection of lines and their applications

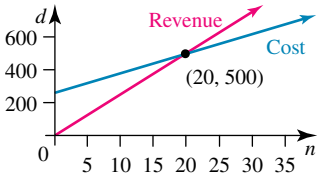
1. (12, 10.5)

2. a. (−0.5, −5) **b.** (8, −3) **c.** (7, −5)

3. a. (−2, 6), (3, 6), (3, −9)

b. 37.5 square units

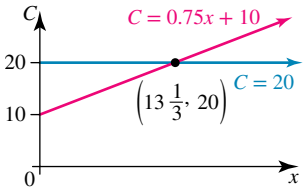
4. a. (20, 500)



b. At least 21 items

5. a. $C_1 = 10 + 0.75x$, $C_2 = 20$ where C is the cost and x the distance

b.



c. $13\frac{1}{3}$ km

d. If the distance is less than $13\frac{1}{3}$ km, Pedal On is cheaper; if the distance exceeds $13\frac{1}{3}$ km, Bikes r Gr8 is cheaper.

6. $m = 6$

7. $a = -1.5, b = -2$

8. $p = 10.5$

9. a. $y = -\frac{px}{5} + \frac{q}{5}, y = \frac{3x}{q} - 5$

b. $p = 0.6, q = -25$

c. $pq \neq -15$

10. (1.39, 5.91)

11. (−1, 2)

12. Sample responses can be found in the worked solutions in the online resources. Point of concurrency at (3, −2)

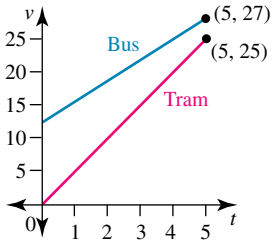
13. $a = 10$

14. a. (−2, −3)

b. Any real number except for $d = 10$

15. a. $s_T = 5t, s_B = 12 + 3t$

b.



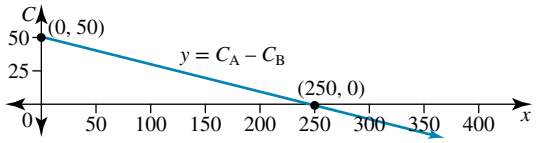
c. After 6 minutes the tram has the faster speed.

16. a. $C_A = 300 + 0.05x$, $C_B = 250 + 0.25x$

b. Cost per kilometre of travel

c. $y = 50 - 0.2x$

d.



e. i. 250 km

ii. More than 250 km

17. a. 6 am (0, 2); 7 am (6, 3); $y = \frac{x}{6} + 2$

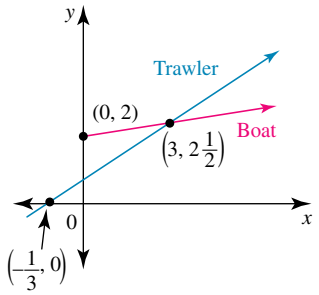
b. Sample responses can be found in the worked solutions in the online resources.

c. (21, 5.5)

d. 6 am $\left(-\frac{1}{3}, 0\right)$; 7 am (1, 1); $y = \frac{3x}{4} + \frac{1}{4}$

e. (3, 2.5)

f. No collision since boat is at common point at 6.30 am and trawler at 8.30 am



18. a. No collision **b.** (4, 11)

Exercise 1.6 Coordinate geometry of the straight line

1. Sample responses can be found in the worked solutions in the online resources.

2. Collinear

3. $b = \frac{5}{7}$

4. $y = x - 2$; not collinear

5. Not collinear so triangle can be formed

6. a. i. $m = 2$ **ii.** $m = -0.5$

b. $m_1 m_2 = -1$

c. $5y + x = 6$

7. a. $\frac{1}{2}$ **b.** $\frac{4}{3}$ **c.** $-\frac{1}{3}$ **d.** -2

8. a. $\frac{4}{5}$ **b.** $\frac{5}{12}$

c. 1 **d.** $0.2 \times -5 = -1$

9. a. $-5x + 7y = 42$ **b.** $20x + 15y = -28$

c. $12x + 8y = -1$ **d.** $x + 3y = 0$

10. (9, 0)

11. a. 56.31° **b.** 116.57° **c.** $y = -x + 9$

12. a. 0.839 **b.** −0.7

13. a. 26.6° **b.** 153.4°

14. a. 60.95° **b.** 143.13° **c.** 90° **d.** 98.13°

15. 78.69°, 75.96°, 2.73°

16. $3x - 2y = -42$

17. a. $y = -1.5x + 6$

b. $2x - 3y = 8$

c. $\frac{26}{3}$ units

18. a. i. $a = -14$ **ii.** $a = 3.5$

b. $c = 2$

c. i. $d = 11.4$ **ii.** $d = 9$ **iii.** $d = 3$

d. $\alpha = 90$

19. $m_{PQ} = m_{SR} = 2, m_{PS} = m_{QR} = 1$
PQRS is a parallelogram as opposite sides are parallel. Adjacent sides are not perpendicular, so PQRS is not a rectangle.

20. $x = 3$

Exercise 1.7 Bisection and lengths of line segments

1. (1.5, 2)

2. a. (5, 3) **b.** (−2, 2)

3. a. (1.5, 2.5) **b.** (20, 12)

4. C

5. P(16, −10)

6. $y = -3x + 7$

7. (4, 3) is midpoint of both AC and BD.

8. $2x + 12y = 77$

9. $y = -2x - 2$

10. $6y - 8x - 29 = 0$

11. $a = 2, b = 5, c = 0$

12. $\sqrt{109} \approx 10.44$

13. 15

14. a. $\sqrt{72} = 6\sqrt{2}$ **b.** $\sqrt{52} = 2\sqrt{13}$

15. BC

16. 10.01

17. $p = \pm 5$

18. a. 10 units **b.** (−10, 6)

c. $4y - 3x = 54$ **d.** $\left(-5, 9\frac{3}{4}\right)$

19. a. 23 units

b. Sample responses can be found in the worked solutions in the online resources.

c. (−3, 4.5)

d. ME = MC = MD = $\sqrt{25.25}$

20. a. (10, 18) **b.** 136π square units

21. Anna

22. a. 4.5 km **b.** $y = -2x + 13$

c. $y = 0.5x + 2.5$ **d.** (4.2, 4.6)

e. 26 minutes

23. a. Perpendicular bisectors are concurrent.

b. (3, 1)

24. a. Medians are concurrent.

b. $\left(\frac{10}{3}, \frac{4}{3}\right)$

1.8 Review: exam practice

Short answer

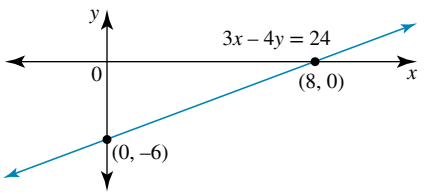
1. a. $x = 0$ **b.** $x = 4$

c. $x = 3$ **d.** $x = 5b$

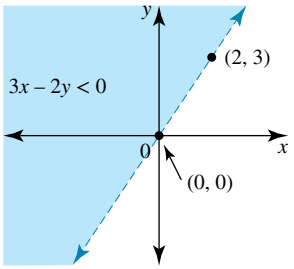
2. $x = 6; y = -4; z = -2$

3. a. $x \geq 1$ **b.** $x < -3$

4. a.



b.



5. a. $7y - 2x = 66$
b. $2y + 7x = 28$
c. $y = 3x - 25$

6. $\sqrt{45}$

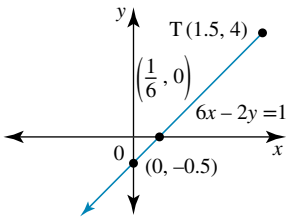
Multiple choice

1. D	**2.** D	**3.** C	**4.** A	**5.** D
6. B	**7.** E	**8.** D	**9.** B	**10.** C

Extended response

1. a. 9 km **b.** 4.27 metres **c.** $3y - 8x = 0$

d. i.



ii. $y = -\frac{x}{3}$; 16 cm

2. a. i. $y = x + 1$; E(9, 10)

ii. $y = -x + 19$; 10 metres

b. Sample responses can be found in the worked solutions in the online resources.

c. i. $r = \frac{90}{k}$ **ii.** $15 \leq k \leq 22.5$

3. a. 25 litres/day

b. t is the independent variable; $V = 1000 + 25t$

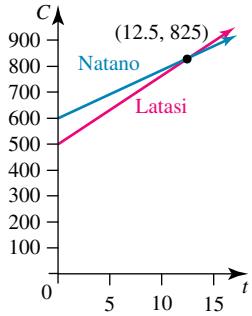
c. Volume is the sum of 1000 litres and an amount directly proportional to the time since 1 April.

d. 950 litres

e. i. $C_L = 500 + 26t$; $C_N = 600 + 18t$; cost is C dollars, construction time is t hours.

ii. 12.5 hours; $825

iii.



iv. Latasi Company is the cheaper.

4. a. $y = \frac{x}{2}$

b. C(4, 2)

c. $y = 3x - 5$

d. E (6, 3)

e. D (4, 7)

f. ABCD is a kite with AD = CD and AB = BC.